OXFORD IB DIPLOMA PROGRAMME

WORKED SOLUTIONS

MATHEMATICS: ANALYSIS AND APPROACHES

HIGHER LEVEL

COURSE COMPANION

ENHANCED ONLINE

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1 From patterns to generalizations: sequences and series

Skills check

1 a $3x+5x-20=20x+4$

$\Rightarrow 8x-20x=20+4$
$\Rightarrow -12x=24$

$x=-2$

b $\frac{x+1}{2x-1}=\frac{x-3}{2x+1}$

$\Rightarrow (x+1)(2x+1)=(x-3)(2x-1)$
$\Rightarrow 2x^2+3x+1=2x^2-7x+3$
$\Rightarrow 10x=2$

$x=\frac{1}{5}$

2 a $\frac{1+\sqrt{2}}{1-\sqrt{2}}=\frac{\left(1+\sqrt{2}\right)^2}{\left(1-\sqrt{2}\right)\left(1+\sqrt{2}\right)}$

$=\frac{1+2\sqrt{2}+2}{1-2}=\frac{3+2\sqrt{2}}{-1}$

$-3-2\sqrt{2}$

b $\frac{2\sqrt{2}}{1-\sqrt{3}}=\frac{2\sqrt{2}\left(1+\sqrt{3}\right)}{\left(1-\sqrt{3}\right)\left(1+\sqrt{3}\right)}$

$=\frac{2\sqrt{2}+2\sqrt{6}}{-2}$

$-\sqrt{2}-\sqrt{6}$

3 $\frac{x}{x+1}-\frac{1}{2x-1}+\frac{2}{x-1}$

$=\frac{x(2x-1)(x-1)-(x+1)(x-1)+2(x+1)(2x-1)}{(x+1)(2x-1)(x-1)}$

$=\frac{x\left(2x^2-3x+1\right)-\left(x^2-1\right)+2\left(2x^2+x-1\right)}{\left(x^2-1\right)(2x-1)}$

$=\frac{2x^3-3x^2+x-x^2+1+4x^2+2x-2}{\left(x^2-1\right)(2x-1)}$

$\frac{2x^3+3x-1}{(x^2-1)(2x-1)}$

Exercise 1A

1 **a** Next three terms are 9, 10.5, 12

The sequence is obtained by adding 1.5 to the previous term and can be written as
$3,\ 3+1.5,\ 3+2(1.5),\ \ldots,\ 3+(n-1)(1.5)$

$u_n = 1.5n + 1.5,\ n \in \mathbb{Z}^+$

b Next three terms are 5, 2, -1

The sequence is obtained by subtracting 3 from the previous term and can be written as
$17,\ 17+(-3),\ 17+2(-3),\ \ldots,\ 17+(n-1)(-3)$

$u_n = 20 - 3n,\ n \in \mathbb{Z}^+$

c Next three terms are 243, 729, 2187

The sequence is obtained by multiplying the previous term by 3 and can be written as
$3,\ 3\times 3,\ 3\times 3^2,\ 3\times 3^3,\ \ldots,\ 3\times 3^{n-1}$

$u_n = 3^n$

d Next three terms are $\frac{13}{16}, \frac{16}{19}, \frac{19}{22}$

The sequence is obtained by adding 3 to both the previous numerator and denominator and can be written as $\frac{1}{4}, \frac{1+3}{4+3}, \frac{1+2(3)}{4+2(3)}, \frac{1+3(3)}{4+3(3)}, \ldots, \frac{1+(n-1)(3)}{4+(n-1)(3)}$

$u_n = \frac{3n-2}{3n+1},\ n \in \mathbb{Z}^+$

e Next three terms are $\frac{1}{90}, \frac{1}{132}, \frac{1}{182}$

The sequence can be written as $\frac{1}{1\times 2}, \frac{1}{3\times 4}, \frac{1}{5\times 6}, \ldots, \frac{1}{(2n-1)(2n)}$

$u_n = \frac{1}{(2n-1)(2n)},\ n \in \mathbb{Z}^+$

2 **a** $u_r = 3 - 2r$

$u_1 = 3 - 2 = 1$
$u_2 = 3 - 2\times 2 = -1$
$u_3 = 3 - 2\times 3 = -3$
$u_4 = 3 - 2\times 4 = -5$
$u_5 = 3 - 2\times 5 = -7$

1, -1, -3, -5, -7

b $u_r = \frac{r}{2r+1}$

$u_1 = \frac{1}{2\times 1+1},\ u_2 = \frac{2}{2\times 2+1},\ u_3 = \frac{3}{2\times 3+1},\ u_4 = \frac{4}{2\times 4+1},\ u_5 = \frac{5}{2\times 5+1}$

$\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}$

c $u_r = 2r + (-1)^r r$

$u_1 = 2 \times 1 + (-1)^1 \times 1 = 1$
$u_2 = 2 \times 2 + (-1)^2 \times 2 = 6$
$u_3 = 2 \times 3 + (-1)^3 \times 3 = 3$
$u_4 = 2 \times 4 + (-1)^4 \times 4 = 12$
$u_5 = 2 \times 5 + (-1)^5 \times 5 = 5$

1, 6, 3, 12, 5

d $u_r = (-1)^r \times 2$

$u_1 = (-1)^1 \times 2 = -2$
$u_2 = (-1)^2 \times 2 = 2$
$u_3 = (-1)^3 \times 2 = -2$
$u_4 = (-1)^4 \times 2 = 2$
$u_5 = (-1)^5 \times 2 = -2$

-2, 2, -2, 2, -2

e $u_r = \frac{3}{2^{r-1}}$

$u_1 = \frac{3}{2^{1-1}} = 3$
$u_2 = \frac{3}{2^{2-1}} = \frac{3}{2}$
$u_3 = \frac{3}{2^{3-1}} = \frac{3}{4}$
$u_4 = \frac{3}{2^{4-1}} = \frac{3}{8}$
$u_5 = \frac{3}{2^{5-1}} = \frac{3}{16}$

$3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{16}$

3 **a** 5, 10, 15, 20, …. The multiples of 5

$\{u_r\} = \{5r\}$, $r \in \mathbb{Z}^+$

b 6, 14, 22, 30, … The sequence is obtained by adding 8 to the previous term and can be written as

$\{u_r\} = \{8r - 2\}$, $r \in \mathbb{Z}^+$

c The sequence is obtained by multiplying the previous term by $\frac{1}{2}$ and can be written as

$$\{u_r\} = \left\{\frac{1}{2^r}\right\},\ r \in \mathbb{Z}^+$$

d The sequence is obtained by multiplying the previous term by $-\frac{1}{3}$ and can be written as

$$\{u_r\} = \left\{\left(-\frac{1}{3}\right)^{r-1}\right\},\ r \in \mathbb{Z}^+$$

e The sequence can be written as $0\times 2,\ 1\times 3,\ 2\times 4,\ 3\times 5,\ \ldots,\ (n-1)\times(n+1)$

OR The sequence can be written as $1^2-1,\ 2^2-1,\ 3^2-1,\ 4^2-1,\ \ldots$

$$\{u_r\} = \{r^2-1\},\ r \in \mathbb{Z}^+$$

4 **a** $\sum_{r=1}^{4} 2r(1-r) = 0-4-12-24$

b $\sum_{r=0}^{5} (-1)^r r^2 = 0-1+4-9+16-25$

c $\sum_{r=1}^{5} \frac{r}{3r-1} = \frac{1}{2}+\frac{2}{5}+\frac{3}{8}+\frac{4}{11}+\frac{5}{14}$

d □ $\sum_{r=1}^{4} 5 = 5+5+5+5$

Explanation: think of this as
$\sum_{r=1}^{4} 5 = \sum_{r=1}^{4} 5r^0$ or $\sum_{r=1}^{4} 5 = \sum_{r=1}^{4} 5\cdot\frac{r}{r}$

e $\sum_{r=0}^{3} (r^2-3) = -3-2+1+6$

5 **a** $\sum_{r=1}^{\infty} \frac{r+1}{r^2} = \frac{1+1}{1^2}+\frac{2+1}{2^2}+\frac{3+1}{3^2}+\frac{4+1}{4^2}+\frac{5+1}{5^2}+\ldots$

$= 2+\frac{3}{4}+\frac{4}{9}+\frac{5}{16}+\frac{6}{25}+\ldots$

b $\sum_{r=1}^{\infty} \frac{(-1)^r}{2r^2-1} = \frac{(-1)^1}{2(1)^2-1}+\frac{(-1)^2}{2(2)^2-1}+\frac{(-1)^3}{2(3)^2-1}+\frac{(-1)^4}{2(4)^2-1}+\frac{(-1)^5}{2(5)^2-1}+\ldots$ □

$= -1+\frac{1}{7}-\frac{1}{17}+\frac{1}{31}-\frac{1}{49}+\ldots$

c $\sum_{r=1}^{20} r(5r-1) = 1(5\times 1-1)+2(5\times 2-1)+3(5\times 3-1)+4(5\times 4-1)+5(5\times 5-1)+\ldots$

$= 4+18+42+76+120+\ldots$

d $\sum_{r=0}^{5} (2^r-3) = (2^0-3)+(2^1-3)+(2^2-3)+(2^3-3)+(2^4-3)+\ldots$

$= -2-1+1+5+13+\ldots$

e $\sum_{r=1}^{\infty} r^r = 1^1 + 2^2 + 3^3 + 4^4 + 5^5 + \ldots = 1 + 4 + 27 + 256 + 3125 + \ldots$

6 **a** The series can be written as $8 + (8-3) + (8-2\times 3) + (8-3\times 3) + (8-4\times 3)$

It has five terms and the general term can be written as $u_r = 11 - 3r$

$\sum_{r=1}^{5}(11-3r)$ □

b The series can be written as $(1\times 3) + (2\times 5) + (3\times 7) + (4\times 9) + (5\times 11)$

It has five terms and the general term can be written as $u_r = r\times(2r+1)$

$\sum_{r=1}^{5} r(2r+1)$

c The series can be written as $\frac{0}{2} + \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \ldots$

It is an infinite series and the general term can be written as $u_r = \frac{r-1}{r+1}$

$\sum_{r=1}^{\infty} \frac{r-1}{r+1}$

d The series can be written as $1^2 + 3^2 + 5^2 + 7^2 + 9^2$

It has five terms and the general term can be written as $u_r = (2r-1)^2$

$\sum_{r=1}^{5}(2r-1)^2$

e The series consists of the multiples of 3k

It has five terms and the general term can be written as $u_r = r\times(3k)$

$\sum_{r=1}^{5} 3kr$

Exercise 1B

1 **a** $u_1 = 3,\ d = 5$

$\therefore u_n = 3 + 5(n-1) = 5n - 2$

b $u_1 = 101,\ d = -4$

$\therefore u_n = 101 - 4(n-1) = 105 - 4n$

c $u_1 = a - 3,\ d = 4$

$\therefore u_n = a - 3 + 4(n-1) = 4n + a - 7$

d $u_1 = -20,\ d = 15$

$\therefore u_n = -20 + 15(n-1) = 15n - 35$

2 **a** $u_1 = 5,\ d = 6$

$\therefore u_{15} = 5 + 6(15-1) = 5 + 6(14) = 89$

b $u_1 = 10,\ d = -7$

$u_{11} = 10 - 7(11-1) = 10 - 7(10) = -60$

c $u_1 = a,\ d = 2$

$u_{17} = a + 2(17-1) = a + 2(16) = a + 32$

d $u_1 = 16,\ d = -4$

$\therefore u_{n+1} = 16 - 4(n+1-1) = 16 - 4n$

3 **a** $u_1 = 16,\ d = -5$
$u_n = 21 - 5n = -64$
$\Rightarrow 5n = 85$
$\Rightarrow n = 17$

b $u_1 = -108,\ d = 7$
$u_n = 7n - 115 = 60$
$\Rightarrow 7n = 175$
$\Rightarrow n = 25$

c $u_1 = -15,\ d = -4$
$u_n = -11 - 4n = -95$
$\Rightarrow 4n = 84$
$\Rightarrow n = 21$

d $u_1 = 2a + 5,\ d = -2$
$u_n = -2n + 2a + 7 = 2a - 23$
$\Rightarrow 2n = 30$
$\Rightarrow n = 15$

4 **a** $u_1 = 5(1) - 7 = -2,$
$u_2 = 5(2) - 7 = 3$
$d = 3 - (-2) = 5$

b $u_1 = 3(1) + 11 = 14,$
$u_2 = 3(2) + 11 = 17,$
$d = 17 - 14 = 3$

c $u_1 = 6 - 11(1) = -5,$
$u_2 = 6 - 11(2) = -16,$
$d = -16 - (-5) = -11$

d $u_1 = 2a + 2(1) + 1 = 2a + 3,$
$u_2 = 2a + 2(2) + 1 = 2a + 5,$
$d = 2a + 5 - (2a + 3) = 2$

5 $u_6 = u_1 + d(6-1) = u_1 + 7(5) = u_1 + 35 = 37$

$\Rightarrow u_1 = 2$
$\therefore u_n = 2 + 7(n-1) = 7n - 5$

6 $u_5 = u_1 + d(5-1) = 0 \Rightarrow u_1 + 4d = 0$

$u_{15} = u_1 + d(15-1) = 180 \Rightarrow u_1 + 14d = 180$
Subtracting the first equation from the second:
$10d = 180 \Rightarrow d = 18$
and substituting this into the first equation,
$u_1 = -4(18) = -72$

7 **Method 1:**

Let the three terms be $a,\ a+d,\ a+2d$

$\Rightarrow a + (a+d) + (a+2d) = 3a + 3d = 24 \Rightarrow a + d = 8$
and $a(a+d)(a+2d) = -640$
Substituting the first equation into the second,
$a(8)(a + 2(8-a)) = -640$
$\Rightarrow 8a(16-a) = -640$
$\Rightarrow 16a - a^2 = -80$
$\Rightarrow a^2 - 16a - 80 = 0$
$\Rightarrow (a-20)(a+4) = 0$ so $a = -4$ or $a = 20$
If $a = -4,\ d = 12$ so the numbers are -4, 8, 20
If $a = 20,\ d = -12$ so the numbers are 20, 8, -4

Method 2:

Let the three terms be $a-d,\ a,\ a+d$

Sum of terms $3a = 24 \Rightarrow a = 8$

Product of terms $a(a^2 - d^2) = -640$

Substitute for a and solve

$8(64 - d^2) = -640$
$\Rightarrow (64 - d^2) = -80$
$\Rightarrow d^2 = 144$
$\Rightarrow d = \pm 12$

Substituting for a and d in $a-d,\ a,\ a+d$ the three numbers would either be $-4, 8, 12$ or $20, 8, -4$

8 In year 2017, Jung Ho earned $38000 + 17(500) = 46500$

$38000 \times 1.5 = 57000$
$\therefore 38000 + 500n \geq 57000$
$\Rightarrow n \geq 38$ so in the year 2038

9 **a** This is an arithmetic series with $u_1 = 3,\ d = -3 - 3 = -6$

$u_n = 9 - 6n = -93$
$\Rightarrow 6n = 102$
$\Rightarrow n = 17$

Using the formula $S_n = \frac{n}{2}(u_1 + u_n)$

$S_{17} = \frac{17}{2}(3 + (-93)) = \frac{17}{2} \times -90 = -765$

b This is an arithmetic series with $u_1 = 31,\ d = 40 - 31 = 9$

$u_n = 9n + 22 = 517$
$\Rightarrow 9n = 495$
$\Rightarrow n = 55$

$\therefore S_{55} = \frac{55}{2}(31 + 517) = \frac{55}{2}(548) = 15070$

c This is an arithmetic series with $u_1 = a - 1,\ d = a + 2 - (a - 1) = 3$

$u_n = (a - 1) + (n - 1) \times 3 = a + 146$
$\Rightarrow a + 3n - 4 = a + 146$
$\Rightarrow 3n = 150$
$\Rightarrow n = 50$

$\therefore S_{50} = \frac{50}{2}(a - 1 + a + 146) = 25(2a + 145) = 50a + 3625$

10 **a** Since $3r - 8$ is linear relation this is an arithmetic series with 50 terms.

$u_1 = 3 - 8 = -5$
$u_{50} = 150 - 8 = 142$
$S_{50} = \frac{50}{2}(-5 + 142) = 3425$

b Since $7 - 8r$ is linear relation this is an arithmetic series with 100 terms.

$u_1 = 7 - 8 = -1$
$u_{100} = 7 - 800 = -793$
$S_{100} = \frac{100}{2}(-1 - 793) = -39700$

c Since $2ar - 1$ is linear relation in r, a is a constant this is an arithmetic series with 20 terms.

$u_1 = 2a - 1$
$u_{20} = 40a - 1$
$S_{20} = \frac{20}{2}(2a - 1 + 40a - 1) = 420a - 20$

11 a This is an arithmetic sequence with $u_1 = 4,\ d = -5$

Using the formula $S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$

$\therefore S_{15} = \frac{15}{2}\left(2 \times 4 - 5 \times 14\right) = -465$

b This is an arithmetic sequence with $u_1 = 3,\ d = 8$

Using the formula $S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$

$\therefore S_{10} = \frac{10}{2}\left(2 \times 3 + 9 \times 8\right) = 390$

c This is an arithmetic sequence with $u_1 = 1,\ d = -5$

Using the formula $S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$

$\therefore S_{20} = \frac{20}{2}\left(2 \times 1 - 5 \times 19\right) = -930$

12 $u_5 = u_1 + 4d = 19$

$u_{10} = u_1 + 9d = 39$
$\therefore u_{10} - u_5 = 5d = 20 \Rightarrow d = 4$
$\Rightarrow u_1 = 19 - 4d = 3$
$\therefore S_{25} = \frac{25}{2}\left(2 \times 3 + 24 \times 4\right) = 1275$

13 a $u_3 = u_1 + 2d = -8$

$S_{10} = \frac{10}{2}\left(2u_1 + 9d\right) = -230 \Rightarrow 2u_1 + 9d = -46$
Multiplying the first equation by 9: $9u_1 + 18d = -72$
Multiplying the second equation by 2: $4u_1 + 18d = -92$
Subtracting: $5u_1 = 20 \Rightarrow u_1 = 4$

b $u_1 = 4 \Rightarrow d = \frac{-8 - u_1}{2} = -6$

$\therefore S_{13} = \frac{13}{2}\left(2 \times 4 - 6 \times 12\right) = -416$

14 $S_1 = 6(1) - 3(1)^2 = 3 \Rightarrow u_1 = 3$

$S_2 = 6(2) - 3(2)^2 = 12 - 12 = 0$
So $S_2 - S_1 = u_2 = -3$
$d = u_2 - u_1 = -3 - 3 = -6$
The first four terms of the sequence are
3, -3, -9, -15

15 $S = 1 + 3 + 5 + \ldots + 299$ There are 150 odd numbers since $2n - 1 = 299 \Rightarrow n = 150$

Using the formula $S_n = \frac{n}{2}(u_1 + u_n)$

$$S_{150} = \frac{150}{2}(1+299) = 22500$$

Exercise 1C

1 **a** $u_5 = 3^4 = 81$

$u_n = 3^{n-1}$

b $u_5 = \frac{1}{2}$

$u_n = 8\left(\frac{1}{2}\right)^{n-1} = 2^3(2)^{1-n} = 2^{4-n} = \frac{1}{2^{n-4}}$

c $u_5 = \frac{x^9}{2}$

$u_n = \frac{x}{2}\left(x^2\right)^{n-1} = \frac{x^{2n-1}}{2}$

d $u_5 = -3$

$u_n = 3(-1)^n$

2 **a** $r = \frac{21}{63} = \frac{1}{3}$

$u_6 = 63\left(\frac{1}{3}\right)^5 = \frac{7}{27}$

b $r = \frac{81}{2 \cdot 243} = \frac{1}{6}$

$u_7 = 243 \cdot \left(\frac{1}{6}\right)^6 = \frac{1}{192}$

c $r = -\frac{a}{6} \cdot \frac{2}{a} = -\frac{1}{3}$

$u_5 = \frac{a}{2}\left(-\frac{1}{3}\right)^4 = \frac{a}{162}$

3 **a** $r = \frac{0.06}{0.02} = 3$

$0.02 \cdot 3^{n-1} = 393.66$
$\Rightarrow 3^{n-1} = 19683$

Using solve or Nsolve (depending on GDC type)

n = 10

b $r = \frac{32}{64} = \frac{1}{2}$

$$64\left(\frac{1}{2}\right)^{n-1} = \frac{1}{128}$$
$$\Rightarrow 2^6 \times 2^{1-n} = 2^{-7}$$
$$\Rightarrow 7 - n = -7$$
$$\Rightarrow n = 14$$

or using technology

4 $u_4 = u_1 r^3 = 6$

$$u_7 = u_1 r^6 = 48$$
$$\therefore \frac{u_1 r^6}{u_1 r^3} = r^3 = \frac{48}{6} = 8 \Rightarrow r = 2$$
$$\Rightarrow u_1 = \frac{6}{2^3} = \frac{3}{4}$$

5 $u_3 = u_1 r^2 = 6$

$$u_5 = u_1 r^4 = 54$$
$$\therefore \frac{u_5}{u_3} = \frac{u_1 r^4}{u_1 r^2} = r^2 = \frac{54}{6} = 9 \Rightarrow r = \pm 3$$
$$\Rightarrow u_1 = \frac{6}{(\pm 3)^2} = \frac{2}{3}$$

$u_6 = u_1 r^5 = \frac{2}{3}(\pm 3)^5 = \pm 162$ depending on which ratio is used

6 $u_1 = 9$

$$u_5 = u_1 r^4 = 9r^4 = 16$$
$$\Rightarrow r^4 = \frac{16}{9} \Rightarrow r = \pm\frac{2}{\sqrt{3}} = \pm\frac{2\sqrt{3}}{3}$$

So two different sequences arise depending on which common ratio is used. In either case, the seventh term is

$$u_7 = u_1 r^6 = 9\left(\pm\frac{2\sqrt{3}}{3}\right)^6 = \frac{64}{3}$$

7 $r = \frac{a+2}{3a+1} = \frac{a-4}{a+2}$

$$\Rightarrow (a+2)^2 = (a-4)(3a+1)$$
$$\Rightarrow a^2 + 4a + 4 = 3a^2 - 11a - 4$$
$$\Rightarrow 2a^2 - 15a - 8 = 0$$
$$\Rightarrow (2a+1)(a-8) = 0$$
$$\Rightarrow a = -\frac{1}{2} \quad \text{or} \quad a = 8$$

If $a = -\frac{1}{2}$, $r = \dfrac{-\frac{1}{2}+2}{3\left(-\frac{1}{2}\right)+1} = -3$

If $a = 8$, $r = \dfrac{2}{5}$

8 $r = \dfrac{a+1}{a-1} = \dfrac{a-2}{a+1}$

$$\Rightarrow (a+1)^2 = (a-2)(a-1)$$
$$\Rightarrow a^2 + 2a + 1 = a^2 - 3a + 2$$
$$\Rightarrow 5a = 1 \Rightarrow a = \frac{1}{5}$$
$$\therefore r = \frac{\frac{1}{5}+1}{\frac{1}{5}-1} = -\frac{3}{2}$$

$$u_1 r^3 = a - 1 = -\frac{4}{5}$$
$$\Rightarrow u_1 = \left(-\frac{2}{3}\right)^3\left(-\frac{4}{5}\right) = \frac{32}{135}$$

9 **a** $r = -\dfrac{1}{3}$

$$\therefore S_6 = 3 \cdot \frac{1-\left(-\frac{1}{3}\right)^6}{1-\left(-\frac{1}{3}\right)} = \frac{182}{81}$$

b $r = \dfrac{4}{8} = \dfrac{1}{2}$

$$\therefore S_{10} = 8 \cdot \frac{1-\left(\frac{1}{2}\right)^{10}}{1-\frac{1}{2}} = \frac{1023}{64}$$

c $r = \dfrac{0.03}{0.1} = 0.3$

$$\therefore S_{15} = 0.1 \cdot \frac{1-(0.3)^{15}}{1-0.3} = 0.143 \ \text{(to 3s.f.)}$$

d $r = -\dfrac{0.03}{0.1} = -0.3$

$$S_{15} = 0.1\frac{1-(-0.3)^{15}}{1-(-0.3)} = 0.0769 \quad (3\text{s.f.})$$

10 a $\displaystyle\sum_{i=1}^{6} 7^{3-i} = 7^2 \cdot \frac{1-\left(\frac{1}{7}\right)^6}{1-\frac{1}{7}} = \frac{19608}{343} = 57.2 \quad (\text{to 3s.f.})$

Or using technology

b $\displaystyle\sum_{i=0}^{n-1} 5\times 10^i = 5\sum_{i=0}^{n-1} 10^i = 5\cdot\frac{10^n-1}{10-1} = \frac{5}{9}\left(10^n-1\right)$

11 $u_1 = 3$

$u_7 = u_1 r^6 = 3r^6 = \dfrac{1}{243}$

$r^6 = \dfrac{1}{729} \Rightarrow r = \pm\dfrac{1}{3}$

Therefore there are two possible common ratios, each corresponding to a different sum to infinity

$r = -\dfrac{1}{3}: \; S_\infty = \dfrac{3}{1-\left(-\frac{1}{3}\right)} = \dfrac{9}{4}$

$r = \dfrac{1}{3}: \; S_\infty = \dfrac{3}{1-\frac{1}{3}} = \dfrac{9}{2}$

12 a $u_1 = S_1 = -\dfrac{3}{2}$

$$u_2 = S_2 - S_1 = \left(-\frac{1}{2}\right)^2 - 1 - \left[\left(-\frac{1}{2}\right)^1 - 1\right] = \frac{3}{4}$$

$$u_3 = S_3 - S_2 = \left(-\frac{1}{2}\right)^3 - 1 - \left[\left(-\frac{1}{2}\right)^2 - 1\right] = -\frac{3}{8}$$

b The terms are in geometric progression with $r = -\dfrac{1}{2}$. To see this in general, note

$$u_n = S_n - S_{n-1} = \left(-\frac{1}{2}\right)^n - 1 - \left[\left(-\frac{1}{2}\right)^{n-1} - 1\right] = \left(-\frac{1}{2}\right)^n - \left(-\frac{1}{2}\right)^{n-1}$$

$$= \left(-\frac{1}{2}\right)^{n-1}\left(-\frac{1}{2}-1\right) = -\frac{3}{2}\left(-\frac{1}{2}\right)^{n-1}$$

i.e. the form of a general term in a geometric progression with first term $-\dfrac{3}{2}$ and common ratio $-\dfrac{1}{2}$

13 $r = \dfrac{u_3}{u_2} = \dfrac{28(1-a)}{28} = 1-a$

$$S_3 = \frac{28}{1-a} + 28 + 28(1-a) = 147$$
$$\Rightarrow \frac{28}{1-a} - 28a = 91$$
$$\Rightarrow 28 - 28a(1-a) = 91(1-a)$$
$$\Rightarrow 28 - 28a + 28a^2 = 91 - 91a$$
$$\Rightarrow 28a^2 + 63a - 63 = 0$$
$$\Rightarrow 4a^2 + 9a - 9 = 0$$
$$\Rightarrow (4a-3)(a+3) = 0$$
so $a = \frac{3}{4}$ or $a = -3$

$|1-a| < 1 \Rightarrow 0 < a < 2$ for convergence

$$\therefore a = \frac{3}{4}$$
$$\Rightarrow r = 1 - a = \frac{1}{4}$$

14 Let the three pieces have lengths u_1, u_2 and u_3

$u_3 = u_1 r^2 = 2u_1 \Rightarrow r^2 = 2 \Rightarrow r = \sqrt{2}$

Since the length of the pieces must sum to 2,

$u_1 + \sqrt{2}u_1 + 2u_1 = (3+\sqrt{2})u_1 = 2$

$$\Rightarrow u_1 = \frac{2}{3+\sqrt{2}} = \frac{2(3-\sqrt{2})}{7}$$

15 $\sum_{i=0}^{\infty} (-1)^i \left(\frac{x}{2}+1\right)^i = 1 - \left(\frac{x}{2}+1\right) + \left(\frac{x}{2}+1\right)^2 - \left(\frac{x}{2}+1\right)^3 + \ldots$

The common ratio is $-\left(\frac{x}{2}+1\right)$

Therefore the series converges when

$$\left|-\left(\frac{x}{2}+1\right)\right| < 1$$
$$\Rightarrow \left|\frac{x}{2}+1\right| < 1$$
$$\Rightarrow -1 < \frac{x}{2} + 1 < 1$$
$$\Rightarrow -2 < x + 2 < 2$$
$$\Rightarrow -4 < x < 0$$
When $x = -0.8$,

$u_0 = 1$ and $r = -0.6$

$$\Rightarrow S_\infty = \frac{1}{1-(-0.6)} = \frac{5}{8}$$

Exercise 1D

1 **a** $220 + 7(10) = 290$

b $S_8 = \frac{8}{2}(220 + 290) = 2040$

c $220+10n=\frac{1}{2}(600-20n)$

$\Rightarrow 20n=80$
$\Rightarrow n=4$
so 2014

2 Let Jane's starting salary be S

Then,
$S(1.015)^{11}=49650$
$\Rightarrow S=\frac{49650}{(1.015)^{11}}=42149.535....$
so Jane's starting salary was €42150 to the nearest euro

3 a $2+2^2+2^3+2^4=30$

b $2+2^2+2^3+2^4+...+2^n>10^6$

The left hand side is a geometric series with first term 2 and common ratio 2

$\Rightarrow \frac{2(2^n-1)}{2-1}>10^6$
$\Rightarrow 2(2^n-1)>10^6$

Using GDC

Answer: 19 generations

4 $S_{10}=\frac{10}{2}(2\times 200+9\times 20)=2900$

so 2.9kg
On the first trial she uses 100g of sugar and on the second
she uses 110g. Thereafter, if the sequence is to become geometric
the common ratio is 1.1
$\therefore 0.1\frac{1.1^n-1}{1.1-1}<1.5$
$\Rightarrow 1.1^n<2.5$
Using GDC $n<9.614$
so 9 trials
In general, the geometric model is not reliable, since if Prisana
were to carry out a large number of trials then the cake will become
excessively sweet (since geometric growth is greater than linear growth)

In fact, the ratio of sugar to flour would eventually become 1 (i.e. the mix
is entirely sugar) in the (albeit unrealistic) case that Prisana carries out the trial
a large number of times

5 a Second: $\sqrt{1^2+1^2}=\sqrt{2}$

Third: $\sqrt{\left(\frac{\sqrt{2}}{2}\right)^2+\left(\frac{\sqrt{2}}{2}\right)^2}=1$

Fourth: $\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}=\frac{1}{\sqrt{2}}$

b $\frac{3}{2}\left(2+\sqrt{2}+1+\frac{1}{\sqrt{2}}+\frac{1}{2}\right)=\frac{3}{2}\left(\frac{7}{2}+\frac{3\sqrt{2}}{2}\right)=\frac{3}{4}\left(7+3\sqrt{2}\right)$

c The length converges to a finite value since the common ratio between two consecutive side lengths is $\frac{1}{\sqrt{2}}<1$.

d Area of triangle $=\frac{1}{2}$base $\times$ height

Required area

$$=\frac{1}{2}\left(1^2+\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2\sqrt{2}}\right)^2+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4\sqrt{2}}\right)^2+\left(\frac{1}{8}\right)^2+\left(\frac{1}{8\sqrt{2}}\right)^2\right)$$

$$=\frac{1}{2}\left(1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\left(\frac{1}{2}\right)^5+\left(\frac{1}{2}\right)^6+\left(\frac{1}{2}\right)^7\right)$$

$$=\frac{1}{2}\left(\frac{1\left(1-\left(\frac{1}{2}\right)^8\right)}{1-\frac{1}{2}}\right)=1-\left(\frac{1}{2}\right)^8=0.996$$

e $S_\infty=\frac{1}{2}\times\left(\frac{1}{1-\frac{1}{2}}\right)=1$

6 **a** Interest 12% pa $\Rightarrow 1\%$ per month

Let the payment per month be x. Interest is compounded monthly

After one month the amount due is

$(1500\times 1.01)-x$

After 2 months the amount due is $((1500\times 1.01)-x)\times 1.01-x=1500(1.01)^2-(1.01)x-x$

After 3 months the amount due is

$\left(1500(1.01)^2-(1.01)x-x\right)\times 1.01-x=1500(1.01)^3-(1.01)^2x-(1.01)x-x$

After 24 months the amount due would be

$1500(1.01)^{24}-(1.01)^{23}x-(1.01)^{22}x-\ldots-x=0$

$$\Rightarrow 1500(1.01)^{24}-x\left(\underbrace{(1.01)^{23}+(1.01)^{22}+\ldots+1}_{\text{Geometric series}}\right)=0$$

$$\Rightarrow 1500(1.01)^{24}-x\left(\frac{1.01^{24}-1}{1.01-1}\right)=0$$

$$\Rightarrow 1500(1.01)^{24}=100x\left(1.01^{24}-1\right)$$

$$\Rightarrow 15(1.01)^{24}=x\left(1.01^{24}-1\right)$$

$$\Rightarrow x=\frac{15(1.01)^{24}}{\left(1.01^{24}-1\right)}$$

Using technology

Monthly payments of $70.61

b Total amount paid

$\$70.61 \times 24 = \$1694.64$

=$1695

7 a $\frac{n}{2}\left(2 \cdot 30 + 6\left(n-1\right)\right) = 570$

$\Rightarrow 60n + 6n\left(n-1\right) = 1140$
$\Rightarrow n^2 + 9n - 190 = 0$
$\Rightarrow \left(n+19\right)\left(n-10\right) = 0$
$\therefore n = 10$

b $3 + 0.95\left(10\right) = 12.5 \Rightarrow 12.5\text{m}$

c $2.4 + 9(0.15) = 3.75\text{m}$

8 a Rapid: $200 + 10\left(0.05\right)\left(200\right) = 300$ so €300

Quick: $200\left(1.035\right)^{10} = 282.11975...$ so €282

Rapid/Quick: $100 + 10\left(0.05\right)\left(100\right) + 100\left(1.035\right)^{10} = 291.0599...$ so €291

b Rapid: $200 + 25\left(0.05\right)\left(200\right) = 450$ so €450

Quick: $200\left(1.035\right)^{25} = 472.649...$ so €473

Rapid/Quick: $100 + 25\left(0.05\right)\left(100\right) + 100\left(1.035\right)^{25} = 461.324...$ so €461

c The investments will be approximately equal when

After n years

Rapid investment: $200 + 10n$

Quick investment: 200×1.034^n

Rapid/Quick : $100 + 5n + 100 \times 1.035^n$

Using tables on GDC:

After 21 years the three investments yield approximately the same amount.

9 a Suppose Karim invested $\$x$ in savings, therefore $\$\left(x + 1000\right)$in bonds

and $\$\left(4000 - 2x\right)$ in shares
$75 = 0.015\left(x\right) + 0.025\left(x + 1000\right) - 0.01\left(4000 - 2x\right)$
$\Rightarrow 90 = 0.06x$
$\Rightarrow x = 1500$
so $1500 in savings, $2500 in bonds and $1000 in shares

b Now Karim is investing $1500 in savings for 10 years,

\$990 in savings for 9 years and \$2500 in bonds for 10 years.
Therefore,

$1500+10(0.015)(1500)+990+9(0.015)(990)+2500(1.025)^{10}=6048.861\ldots$

so \$6048.86 =\$6049 to the nearest dollar

c $2500(1.025)^{10}+2500+10(0.015)(2500)-6048.86136\ldots=26.3500\ldots$

so \$26

10 a $x(1+0.375+0.375^2+0.375^3)$, where x is the amount administered each time.

b $x(1+0.375+0.375^2+\ldots+0.375^{39})\leq 8$

$\Rightarrow x\left(\dfrac{1-0.375^{40}}{1-0.375}\right)\leq 8$

$\Rightarrow x\leq\dfrac{8(1-0.375)}{(1-0.375^{40})}$

5 mg should be administered each time.

c The amount of medication in the bloodsteam after n administrations is given by

$5\left(\dfrac{1-0.375^n}{1-0.375}\right)=7$

$\Rightarrow 1-0.375^n=\dfrac{7(1-0.375)}{5}$

$\Rightarrow 0.375^n=1-\dfrac{7(1-0.375)}{5}$

Using technology to solve:

There are 7mg/ml drug in the bloodstream after the third administration.

Exercise 1E

1 $(a+b)^2+(a-b)^2=(a^2+2ab+b^2)+(a^2-2ab+b^2)=2a^2+2b^2=2(a^2+b^2)$

2 A general odd number can be written in the form $2k+1$ with k $\in\mathbb{Z}$

$\therefore$ Consider two general odd numbers $2n+1$ and $2m+1$, $n,m\in\mathbb{Z}$
Then,
$(2n+1)(2m+1)=4nm+2n+2m+1=2(2nm+n+m)+1=2p+1$
$p=2nm+n+m\in\mathbb{Z}$
$\therefore$ $2p+1$ is an odd number

3 A four digit number represented by $a_3a_2a_1a_0$ (not to be confused with a product)

can be written in the form

$N=a_3\times10^3+a_2\times10^2+a_1\times10+a_0$

You are given that $a_3+a_2+a_1+a_0=9m$, $m\in\mathbb{Z}^+$

$\therefore N = (999+1)a_3 + (99+1)a_2 + (9+1)a_1 + a_0$
$= (999a_3 + 99a_2 + 9a_1) + (a_3 + a_2 + a_1 + a_0)$
$= 9(111a_3 + 11a_2 + a_1) + 9m$
$= 9(111a_3 + 11a_2 + a_1 + m)$
i.e. if 9 divides the sum of the digits the number itself is divisible by 9
Hence 3978, 9864 and 5670 are divisible by 9 but 5453 and 7898 are not

4 $(ad+bc)^2 + (bd-ac)^2$

$= a^2d^2 + 2abcd + b^2c^2 + b^2d^2 - 2abcd + a^2c^2$
$= a^2d^2 + b^2c^2 + b^2d^2 + a^2c^2$
$= a^2(c^2+d^2) + b^2(c^2+d^2)$
$= (a^2+b^2)(c^2+d^2)$

5 $S = \frac{1}{3} - \frac{2}{9} + \frac{1}{27} - \frac{2}{81} + \frac{1}{243} - \frac{2}{729} + \ldots$

$S = \frac{1}{3} + \frac{1}{27} + \frac{1}{243} + \ldots - \frac{2}{9} - \frac{2}{81} - \frac{2}{729} - \ldots$

$S = \frac{1}{3} + \frac{1}{3}\left(\frac{1}{3}\right)^2 + \frac{1}{3}\left(\frac{1}{3}\right)^4 + \ldots - 2\left(\frac{1}{9} + \frac{1}{9}\left(\frac{1}{9}\right) + \frac{1}{9}\left(\frac{1}{9}\right)^2 + \ldots\right)$

Two different infinite geometric series, each with common ratio $\frac{1}{9}$,

and so both series converge.

$S = \left(\dfrac{\frac{1}{3}}{1-\frac{1}{9}}\right) - 2\left(\dfrac{\frac{1}{9}}{1-\frac{1}{9}}\right)$

$= \left(\frac{1}{3} \times \frac{9}{8}\right) - 2\left(\frac{1}{9} \times \frac{9}{8}\right) = \frac{1}{8}$

6 Consider an arbitrary integer $n \in \mathbb{Z}$. Then,

$(n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n+1$ is odd

7 $\frac{1}{n-1} - \frac{1}{n} + \frac{1}{n+1}$

$= \dfrac{n(n+1) - (n-1)(n+1) + n(n-1)}{n(n-1)(n+1)}$

$= \dfrac{n^2 + n - (n^2-1) + n^2 - n}{n(n^2-1)}$

$= \dfrac{n^2+1}{n(n^2-1)}$

$\therefore \frac{1}{5} - \frac{1}{6} + \frac{1}{7} = \dfrac{6^2+1}{6(6^2-1)} = \dfrac{37}{6(35)} = \dfrac{37}{210}$

8 Area of trapezium: $\frac{a+b}{2}h = \frac{a+b}{2}(a+b)$

Similarly, the area in terms of the triangles BAE, BEC and EDC are

$\frac{1}{2}ab + \frac{1}{2}c^2 + \frac{1}{2}ab = ab + \frac{1}{2}c^2$

Equating the areas,

$\frac{(a+b)^2}{2} = ab + \frac{1}{2}c^2 \Rightarrow (a+b)^2 = 2ab + c^2$

$\Rightarrow a^2 + 2ab + b^2 = 2ab + c^2$

$\Rightarrow a^2 + b^2 = c^2$

Exercise 1F

1 Suppose for the sake of contradiction that n^2 is odd but n is even

Then $n^2 = 2m + 1$ for some $m \in \mathbb{Z}$ and $n = 2k$ for some $k \in \mathbb{Z}$

But then $n^2 = (2k)^2 \Rightarrow 4k^2 = 2m + 1$

$4k^2$ is even but $(2m+1)$ is odd, so this is a contradiction

$\therefore n^2$ is odd $\Rightarrow n$ is also odd

2 Assume for the sake of contradiction that $\sqrt{3} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$

are coprime (i.e. they have no common factors).

Then, $3 = \frac{a^2}{b^2} \Rightarrow a^2 = 3b^2$

If p is a prime number and p divides a^2, where $a \in \mathbb{Z}^+$, then p must divide a.
Therefore, a must be a multiple of 3 $\Rightarrow a = 3k$ for some
$k \in \mathbb{Z}$. This implies $9k^2 = 3b^2 \Rightarrow b^2 = 3k^2$ so b is also divisible
by 3.
Therefore 3 is a common factor of a and b. But we assumed that
a and b have no common factors, so this is a contradiction.

3 Suppose for the sake of contradiction that $\sqrt[5]{2}$ is rational

Then $\sqrt[5]{2}$ can be written in the form $\sqrt[5]{2} = \frac{a}{b}$ where

$a, b \in \mathbb{Z}^+$ are relatively coprime (i.e. share no common factors)

$\therefore a^5 = 2b^5$ so 2 divides $a \Rightarrow a = 2m$ for some $m \in \mathbb{Z}^+$

$\Rightarrow b^5 = 2^4m^5$ so b^5 is even which means that b is also even.

So 2 divides both a and b, but it was assumed that a and b
shared no common factors. This is a contradiction.

4 Suppose for the sake of contradiction that there exist $p, q \in \mathbb{Z}$

such that $p^2 - 8q - 11 = 0$

$\Rightarrow p^2 = 8q + 11$ so p is an odd integer

$\therefore p = 2k + 1$ for some $k \in \mathbb{Z}$

$\therefore (2k+1)^2 = 8q + 11$

$\Rightarrow 4k^2 + 4k + 1 = 8q + 11$

$\Rightarrow 4(k^2 + k - 2q) = 10$

$\Rightarrow 2(k^2 + k - 2q) = 5$

but LHS is even whereas RHS is odd; this is a contradiction

5 Suppose for the sake of contradiction that for some $a, b \in \mathbb{Z}$, $12a^2 - 6b^2 = 0$

$\therefore 12a^2 = 6b^2 \Rightarrow 2a^2 = b^2 \Rightarrow 2 = \frac{a^2}{b^2} = \left(\frac{a}{b}\right)^2 \Rightarrow \sqrt{2} = \frac{a}{b},$

a contradiction since we know that $\sqrt{2}$ is irrational.

6 Suppose for the sake of contradiction that for $a, b, c \in \mathbb{Z}$, the equation $a^2 + b^2 = c^2$ is satisfied

You are given that $a^2 + b^2 = c^2$, where $a, b, c \in \mathbb{Z}$ and $c = 2k + 1$, $k \in \mathbb{Z}$

We are required to prove that either a or b must be even.

Assume that both a and b are odd

$a = 2p + 1$ and $b = 2q + 1$, $p, q \in \mathbb{Z}$
$\Rightarrow a^2 + b^2 = (2p + 1)^2 + (2q + 1)^2$
$= 4p^2 + 4p + 1 + 4q^2 + 4q + 1$
$= 2(2p^2 + 2p + 2q^2 + 2q + 1) = 2n$, $n \in \mathbb{Z}$
You know that $a^2 + b^2 = c^2$ and $c = 2k + 1$, $k \in \mathbb{Z}$
$c^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2m + 1$, $m \in \mathbb{Z}$
$a^2 + b^2 = c^2$
$\Rightarrow 2n = 2m + 1$

The left-hand side is an even number and the right-hand side represents an odd number.

This is a contradiction.

Now let us assume that both a and b are even $a = 2p$ and $b = 2q$

$a^2 + b^2 = (2p)^2 + (2q)^2 = 2(2p^2 + 2q^2) = 2s$, $s \in \mathbb{Z}$
$a^2 + b^2 = c^2$
$\Rightarrow 2s = 2m + 1$

The left-hand side is an even number and the right-hand side represents an odd number which is a contradiction

Hence, we have proved that precisely one of a or b must be even.

7 Suppose there exists $n, k \in \mathbb{Z}$ such that $n^2 + 2 = 4k$

Then n must be divisible by 2 and can be written in the form
$n = 2m$ with $m \in \mathbb{Z}$
$\therefore 4m^2 + 2 = 4k$
$\Rightarrow m^2 - k = -\frac{1}{2}$
But the left-hand side is an integer whereas the right-hand side is
not; this is a contradiction

8 Suppose p is irrational, q is rational and for the sake of contradiction that

$p+q$ is rational. Then,

$q=\frac{a}{b}$ and $p+q=\frac{c}{d}$ for some $a,b,c,d\in\mathbb{Z}$

$\Rightarrow p=\frac{c}{d}-q=\frac{c}{d}-\frac{a}{b}=\frac{bc-ad}{bd}\in\mathbb{Q}$

But by assumption, p was irrational. This is a contradiction.

9 Let $m,n\in\mathbb{Z}^+$ and suppose for the sake of contradiction that $m^2-n^2=1$

Then,

$m^2-n^2=(m+n)(m-n)=1$

Since $m,n\in\mathbb{Z}^+$, $m>n$

The product of two positive integers can only give 1, if both are 1 or both are -1.

i.e.

$m+n=m-n\Rightarrow n=-n$

This is a contradiction since $n\in\mathbb{Z}^+$

10 a Take $m=n=1$

b Take any prime number: the number is certainly divisible by itself

but is still a prime

c Take $n=4$: $2^4-1=16-1=15=(3)(5)$

d Take the same example as in part c.

e $1+2+3=6$, not divisible by 4

f $1+2+3+4=10$, not divisible by 4

Exercise 1G

1 a i $1+3+1$ $\quad 1+3+5+3+1$ $\quad 1+3+5+7+5+3+1$

ii $1+4$ $\quad 4+9$ $\quad 9+16$

b based on line divisions

$1+3+5+7+9+7+5+3+1$ $\quad 1+3+5+7+9+11+9+7+5+3+1$

based on colour $\quad 16+25$ $\quad 25+36$

c Organizing our findings

$1+3+1 \quad = 1+4$

$1+3+5+3+1 \quad = 4+9$

$1+3+5+7+5+3+1 = 9+16$

$1+3+5+7+9+7+5+3+1 \quad = 16+25$

$1+3+5+7+9+11+9+7+5+3+1 = 25+36$

.

.

.

$2\left(1+3+5+\ldots+2k-1\right)+2k+1 = k^2+(k+1)^2$

Conjecture: P(n): $2\left(1+3+5+\ldots+2n-1\right)+2n+1 = n^2+(n+1)^2,\ n \in \mathbb{Z}^+$

d LHS $= 2\left(\underbrace{1+3+5+\ldots+2n-1}_{\text{sum of first n odd numbers}}\right)+2n+1$

$$= 2\left(\frac{n}{2}(1+(2n-1))\right)+2n+1$$
$$= n(2n)+2n+1$$
$$= 2n^2+2n+1$$
$$= n^2+n^2+2n+1$$
$$= n^2+(n+1)^2$$

e P(n): $2\left(1+3+5+\ldots+2n-1\right)+2n+1 = n^2+(n+1)^2,\ n \in \mathbb{Z}^+$

When $n=1$

LHS $= 2(1)+3 = 5$

RHS$= 1^2+2^2 = 5$

LHS=RHS therefore P(1) is true.

Assume that P(k) is true for some $k \in \mathbb{Z}^+$

i.e. $2\left(1+3+5+\ldots+2k-1\right)+2k+1 = k^2+(k+1)^2$

Required to prove that P(k+1) is true

i.e. $2\left(1+3+5+\ldots+(2k-1)+(2k+1)\right)+2k+3 = \left(k+1\right)^2+(k+2)^2$ using the assumption

LHS$= = 2\left(1+3+5+\ldots+(2k-1)\right)+2(2k+1)+2k+3$

$$= 2\left(1+3+5+\ldots+(2k-1)\right)+(2k+1)+4k+4$$
$$= k^2+(k+1)^2+4k+4$$
$$= (k+1)^2+k^2+4k+2$$
$$= \left(k+1\right)^2+(k+2)^2$$

Since P(1) was shown to be true, and it was shown that if P(k) is true, where $k \in \mathbb{Z}^+$, then P(k+1) is true, it follows by the principle of mathematical induction that P(n) is true for all $n \in \mathbb{Z}^+$

2 a $P(n): 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{3}n(n+1)\left(n+\frac{1}{2}\right)$

When $n = 1$,

$\text{LHS} = 1^2 = 1$

$\text{RHS} = \frac{1}{3}1(1+1)(1)\left(1+\frac{1}{2}\right) = \frac{1}{3}(2)\left(\frac{3}{2}\right) = 1$

LHS = RHS $\therefore$ P(1) is true.

Assume the statement is true for $n = k$, where $k \in \mathbb{Z}^+$

Required to prove that when $n = k+1$, $1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{1}{3}(k+1)(k+2)\left(k+\frac{3}{2}\right)$

$$\text{LHS} = \underbrace{1^2 + 2^2 + 3^2 + \ldots + k^2} + (k+1)^2$$

$$= \frac{1}{3}k(k+1)\left(k+\frac{1}{2}\right) + (k+1)^2$$

$$= (k+1)\left(\frac{1}{3}k\left(k+\frac{1}{2}\right) + (k+1)\right)$$

$$= \frac{1}{3}(k+1)\left(k\left(k+\frac{1}{2}\right) + 3(k+1)\right)$$

$$= \frac{1}{3}(k+1)\left(k^2 + \frac{k}{2} + 3k + 3\right)$$

$$= \frac{1}{3}(k+1)\left(k^2 + \frac{7k}{2} + 3\right)$$

$$= \frac{1}{3}(k+1)\left(\frac{2k^2 + 7k + 6}{2}\right)$$

$$= \frac{1}{3}(k+1)\left(\frac{(k+2)(2k+3)}{2}\right)$$

$$= \frac{1}{3}(k+1)(k+2)\left(\frac{(2k+3)}{2}\right)$$

$$= \frac{1}{3}(k+1)(k+2)\left(k+\frac{3}{2}\right)$$

=RHS

Since it was shown that P(1) is true and that P(k + 1) is true given P(k) is true for $k \in \mathbb{Z}^+$ it follows by the principle of mathematical induction that P(n) is true for all $n \in \mathbb{Z}^+$

b $P(n):\ 1 - 4 + 9 - 16 + \ldots + (-1)^{n+1} n^2 = (-1)^{n+1} \frac{n(n+1)}{2}$

When $n = 1$

LHS = 1

RHS$=(-1)^{1+1}\dfrac{1(1+1)}{2}=1$

Assume the statement $\mathrm{P}(k)$ is true for some $k \in \mathbb{Z}^+$ i.e.

$1-4+9-16+\ldots+(-1)^{k+1}k^2=(-1)^{k+1}\dfrac{k(k+1)}{2}$

When $n=k+1$,

LHS $=\underbrace{1-4+9-16+\ldots+(-1)^{k+1}k^2}_{\text{Use assumption}}+(-1)^{k+2}(k+1)^2$

$=(-1)^{k+1}\dfrac{k(k+1)}{2}+(-1)^{k+2}(k+1)^2$

$=(-1)^{k+1}(k+1)\left[\dfrac{k}{2}-(k+1)\right]$

$=(-1)^{k+1}(k+1)\left[\dfrac{k-2(k+1)}{2}\right]$

$=(-1)^{k+1}(k+1)\left(\dfrac{-k-2}{2}\right)$

$=(-1)^{k+2}\dfrac{(k+1)((k+1)+1)}{2}$

i.e. $P(k)\Rightarrow P(k+1)$

Since $P(1)$ is true and $P(k)\Rightarrow P(k+1)$ for $k \in \mathbb{Z}^+$ then by the principle of mathematical induction, the statement is true for all positive integers

c $P(n)$: $\displaystyle\sum_{i=0}^{n}2^i=2^{n+1}-1$

When $n=0$

LHS$=\displaystyle\sum_{i=0}^{0}2^i=2^0=1$

RHS$=2^{0+1}-1=2-1=1$

LHS$=$RHS $\therefore P(1)$ is true

Assume that $P(k)$ is true for some $k \in \mathbb{N}$ i.e. $\displaystyle\sum_{i=0}^{k}2^i=2^{k+1}-1$

When $n=k+1$

$\displaystyle\sum_{i=0}^{k+1}2^i=\sum_{i=0}^{k}2^i+2^{k+1}=\left(2^{k+1}-1\right)+2^{k+1}=2^{k+1}+2^{k+1}-1=2\cdot 2^{k+1}-1=2^{k+2}-1$

i.e. $P(k)\Rightarrow P(k+1)$

Since $P(1)$ is true and $P(k)\Rightarrow P(k+1)$ for $k \in \mathbb{N}$ then by the principle of mathematical induction, the statement is true for all natural numbers

d $P(n)$: 9^n-1 is divisible by 8 (for $n \in \mathbb{N}$)

$P(n): 9^n - 1 = 8A$, for $n \in \mathbb{N}$, $A \in \mathbb{Z}$

When $n = 0$

LHS $= 9^0 - 1 = 0 \;\; = 8 \times 0$

$\therefore \; P(1)$ is true

Assume $P(k)$ to be true for some $k \in \mathbb{N}$

i.e. 8 divides $9^k - 1 \Rightarrow 9^k - 1 = 8m$ for some $m \in \mathbb{Z}$

Then,

$9^{k+1} - 1 = 9 \cdot 9^k - 1 = 9(8m+1) - 1 = 9(8m) + 9 - 1$

$= 8(9m) + 8 = 8(9m+1)$

so 8 also divides $9^{k+1} - 1$

i.e. $P(k) \Rightarrow P(k+1)$

Since $P(0)$ is true and $P(k) \Rightarrow P(k+1)$ for $k \in \mathbb{N}$ then by the principle of mathematical induction, the statement is true for all natural numbers

e $P(n)$: $1^3 + 2^3 + 3^3 + \ldots + n^3 = \dfrac{n^2(n+1)^2}{4}$

LHS $= 1^3 = 1$

RHS $= \dfrac{1^2(1+1)^2}{4} = 1$

LHS = RHS

$\therefore P(1)$ is true

Assume $P(k)$ is true for some $k \in \mathbb{Z}^+$

i.e. $1^3 + 2^3 + 3^3 + \ldots + k^3 = \dfrac{k^2(k+1)^2}{4}$

Then,

$$\underbrace{1^3 + 2^3 + \ldots + k^3}_{\text{use assumption}} + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= \frac{(k+1)^2}{4}\left(k^2 + 4(k+1)\right) = \frac{(k+1)^2\left(k^2 + 4k + 4\right)}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{4} = \frac{(k+1)^2\left((k+1)+1\right)^2}{4}$$

i.e. $P(k) \Rightarrow P(k+1)$

Since $P(1)$ is true and $P(k) \Rightarrow P(k+1)$ for $k \in \mathbb{Z}^+$ then by the principle of mathematical induction, the statement is true for all positive integers.

f $P(n)$: $n^3 - n = 3A$, for $n \in \mathbb{Z}$, $A \in \mathbb{Z}$

When $n=0$:
$1^0-1=0 = 3\times 0$
$\therefore$ The statement $P(0)$ is true
Assume $P(k)$ is true for some $k\in\mathbb{N}$
$k^3-k=3m$ for some $m\in\mathbb{Z}$
$\Rightarrow k^3=3m+k$
When $n=k+1$,
LHS $=(k+1)^3-(k+1)=(k^3+3k^2+3k+1)-(k+1)$
$=3m+3(k^2+k)=3(m+k^2+k),\ m+k^2+k\in\mathbb{Z}$
i.e. $P(k)\Rightarrow P(k+1)$

Since $P(0)$ is true and $P(k)\Rightarrow P(k+1)$ for $k\in\mathbb{N}$ then by the principle of mathematical induction, the statement is true for all natural numbers

g $P(n):\ \dfrac{1}{1\times 2}+\dfrac{1}{2\times 3}+\dfrac{1}{3\times 4}+\ldots+\dfrac{1}{n(n+1)}=\dfrac{n}{n+1}$,

When $n=1$:
LHS$=\dfrac{1}{1\times 2}=\dfrac{1}{2}$
RHS$=\dfrac{1}{1+1}=\dfrac{1}{2}$
LHS=RHS $\therefore P(1)$ is true
Assume $P(k)$ is true for some $k\in\mathbb{Z}^+$
i.e. $\dfrac{1}{1\times 2}+\dfrac{1}{2\times 3}+\ldots+\dfrac{1}{k(k+1)}=\dfrac{k}{k+1}$

When $n=k+1$,
$$\text{LHS}=\underbrace{\frac{1}{1\times 2}+\frac{1}{2\times 3}+\ldots+\frac{1}{k(k+1)}}_{\text{use assumption}}+\frac{1}{(k+1)(k+2)}$$
$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}=\frac{1}{k+1}\left(k+\frac{1}{k+2}\right)$$
$$=\frac{1}{k+1}\left(\frac{k(k+2)+1}{k+2}\right)=\frac{1}{k+1}\left(\frac{k^2+2k+1}{k+2}\right)$$
$$=\frac{1}{k+1}\left(\frac{(k+1)^2}{k+2}\right)=\frac{k+1}{k+2}=\frac{k+1}{(k+1)+1}$$
i.e. $P(k)\Rightarrow P(k+1)$

Since $P(1)$ is true and $P(k)\Rightarrow P(k+1)$ for $k\in\mathbb{Z}^+$ then by the principle of mathematical induction, the statement is true for all positive integers

h $P(n):\ n^3-n=6A$ for all $n\in\mathbb{Z}^+, A\in\mathbb{Z}$

When $n = 1$

$1^3 - 1 = 0 = 0 \times 6$

$\therefore P(1)$ is true

Assume $P(k)$ is true for some $k \in \mathbb{Z}^+$

$k^3 - k = 6m$ for some $m \in \mathbb{Z}$

$\Rightarrow k^3 = k + 6m$

When $n = k + 1$,

$(k+1)^3 - (k+1) = (k^3 + 3k^2 + 3k + 1) - (k+1)$

$= k + 6m + 3k^2 + 2k$

$= 6m + 3k(k+1)$

but $k(k+1)$ must be an even number since any pair of consecutive natural numbers contains an even number

$\therefore k(k+1) = 2r$ for some $r \in \mathbb{Z}^+$

$\Rightarrow (k+1)^3 - (k+1) = 6(m+r)$ which is divisible by 6

i.e. $P(k) \Rightarrow P(k+1)$

Since $P(1)$ is true and $P(k) \Rightarrow P(k+1)$ for $k \in \mathbb{Z}^+$ then by the principle of mathematical induction, the statement is true for all positive integers

i $P(n)$: $2^{n+2} + 3^{2n+1} = 7A \ (n \in \mathbb{Z}^+, A \in \mathbb{Z})$

When $n = 1$

LHS$=2^{1+2} + 3^{2+1} = 2^3 + 3^3 = 8 + 27 = 35 = 7 \times 5$

$\therefore P(1)$ is true

Assume that $P(k)$ is true for some $k \in \mathbb{Z}^+$

$2^{k+2} + 3^{2k+1} = 7m$ for some $m \in \mathbb{Z}^+$

$\Rightarrow 2^{k+2} = 7m - 3^{2k+1}$

When $n = k + 1$,

LHS $=2^{(k+1)+2} + 3^{2(k+1)+1} = 2 \cdot 2^{k+2} + 9 \times 3^{2k+1}$

$= 2(7m - 3^{2k+1}) + 9 \times 3^{2k+1}$

$= 14m - 2 \times 3^{2k+1} + 9 \times 3^{2k+1}$

$= 14m + 7 \cdot 3^{2k+1}$

$= 7(2m + 3^{2k+1})$ where $2m + 3^{2k+1} \in \mathbb{Z}$

so $P(k) \Rightarrow P(k+1)$

Since $P(1)$ is true and $P(k) \Rightarrow P(k+1)$ for $k \in \mathbb{Z}^+$ then by the principle of mathematical induction, the statement is true for all positive integers.

j $P(n)$: $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$

When $n=1$

LHS $=1^2=1$

RHS $=\dfrac{1(2-1)(3)}{3}=1$

LHS $=$RHS

$\therefore P(1)$ is true

Assume that $P(k)$ is true for some $k \in \mathbb{Z}^+$

i.e. $1^2+3^2+5^2+\ldots+(2k-1)^2=\dfrac{k(2k-1)(2k+1)}{3}$

When $n=k+1$

LHS $=\underbrace{1^2+3^2+5^2+\ldots+(2k-1)^2}_{\text{use assumption}}+(2k+1)^2$

$=\dfrac{k(2k-1)(2k+1)}{3}+(2k+1)^2$

$=\dfrac{(2k+1)}{3}\left(k(2k-1)+3(2k+1)\right)$

$=\dfrac{(2k+1)}{3}\left(2k^2+5k+3\right)$

$=\dfrac{(2k+1)(2k+3)(k+1)}{3}$

$=\dfrac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$

i.e. $P(k) \Rightarrow P(k+1)$

Since $P(1)$ is true and $P(k) \Rightarrow P(k+1)$ for $k \in \mathbb{Z}^+$ then by the principle of mathematical induction, the statement is true for all positive integers.

k $P(n)$: $\displaystyle\sum_{r=1}^{n} r(r+1)=\frac{n}{3}(n+1)(n+2)$

When $n = 1$

LHS $=\displaystyle\sum_{r=1}^{1} r(r+1)=1(1+1)=2$

RHS $=\dfrac{1}{3}(1+1)(1+2)=2$

$\therefore P(1)$ is true

Assume $P(k)$ to be true for some $k \in \mathbb{Z}^+$

i.e. $\displaystyle\sum_{r=1}^{k} r(r+1)=\frac{k}{3}(k+1)(k+2)$

When $n = k+1$,

LHS $=\displaystyle\sum_{r=1}^{k+1} r(r+1)=\sum_{r=1}^{k} r(r+1)+(k+1)(k+2)$

$=\dfrac{k}{3}(k+1)(k+2)+(k+1)(k+2)$

$=\dfrac{(k+1)(k+2)}{3}(k+3)=\dfrac{(k+1)((k+1)+1)((k+1)+2)}{3}$

i.e. $P(k) \Rightarrow P(k+1)$

Since $P(1)$ is true and $P(k) \Rightarrow P(k+1)$ for $k \in \mathbb{Z}^+$ then by the principle of mathematical induction, the statement is true for all positive integers.

I $P(n):\ \sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$

When $n = 1$

LHS $= \sum_{r=1}^{1} \frac{1}{r(r+1)} = \frac{1}{1(1+1)} = \frac{1}{2}$

RHS $= \frac{1}{1+1} = \frac{1}{2}$

$\therefore P(1)$ is true

Assume $P(k)$ is true for some $k \in \mathbb{Z}^+$

i.e. $\sum_{r=1}^{k} \frac{1}{r(r+1)} = \frac{k}{k+1}$

When $n = k+1$

LHS $= \sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \sum_{r=1}^{k} \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+2)}$

$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$

$= \frac{1}{k+1}\left(k + \frac{1}{k+2}\right)$

$= \frac{1}{k+1}\left(\frac{k(k+2)+1}{k+2}\right)$

$= \frac{1}{k+1}\left(\frac{k^2+2k+1}{k+2}\right)$

$= \frac{1}{k+1}\left(\frac{(k+1)^2}{k+2}\right) = \frac{k+1}{k+2}$

$\therefore P(k) \Rightarrow P(k+1)$

Since $P(1)$ is true and $P(k) \Rightarrow P(k+1)$ for $k \in \mathbb{Z}^+$ then by the principle of mathematical induction, the statement is true for all positive integers.

3 a Best proved by direct argument:

$(4n+3)^2 - (4n-3)^2$

$= (4n+3+4n-3)(4n+3-4n+3)$

$= (8n)(6) = 48n = 12(4n)$ so is always divisible by 12

(induction amongst other methods is also valid)

b False: substituting $n = 1$ gives 75 which is not prime

c Best proved by induction:

$P(n):\ 1^3 + 3^3 + \ldots + (2n-1)^3 = n^2(2n^2-1)$

When $n = 1$

LHS$=\ 1^3 = 1$

RHS$=1^2(2 \cdot 1^2 - 1) = 1$

LHS=RHS

$\therefore P(1)$ is true:

Assume the statement $P(k)$ is true for some $k \in \mathbb{Z}^+$

i.e. $1^3 + 3^3 + \ldots + (2k-1)^3 = k^2(2k^2-1)$

When $n = k+1$

LHS $= \underbrace{1^3 + 3^3 + \ldots + (2k-1)^3}_{\text{Use assumption}} + (2k+1)^3$

$= k^2(2k^2-1) + (2k+1)^3$

$= 2k^4 - k^2 + 8k^3 + 12k^2 + 6k + 1$

$= 2k^4 + 8k^3 + 11k^2 + 6k + 1$ (use factor theorem to factorize or expand right hand side of P(k+1) to obtain same polynomial)

$= (k+1)(2k^3 + 6k^2 + 5k + 1)$

$= (k+1)(k+1)(2k^2 + 4k + 1)$

$= (k+1)^2\left(2(k+1)^2 - 1\right)$

so $P(k) \Rightarrow P(k+1)$

Since $P(1)$ is true and $P(k) \Rightarrow P(k+1)$ for $k \in \mathbb{Z}^+$ then by the principle of mathematical induction, the statement is true for all positive integers.

d Best proved by induction:

$P(n)$: $1\times2 + 2\times3 + 3\times4 \ldots + (n-1)\times n = \dfrac{n(n^2-1)}{3}$

When $n = 1$

LHS= $0\times1 = 0$

RHS= $\dfrac{1(1^2-1)}{3} = 0$

LHS=RHS

$\therefore P(1)$ is true

Assume the statement $P(k)$ is true for some $k \in \mathbb{Z}^+$

i.e. $1\times2 + 2\times3 + 3\times4 \ldots + (k-1)\times k = \dfrac{k(k^2-1)}{3}$

When $n = k+1$

LHS= $\underbrace{1\times2 + 2\times3 + 3\times4 \ldots + (k-1)\times k}_{\text{use assumption}} + k(k+1)$

$= \dfrac{k(k^2-1)}{3} + k(k+1)$

$= \dfrac{k(k-1)(k+1) + 3k(k+1)}{3}$

$= \dfrac{(k+1)(k(k-1)+3k)}{3}$

$= \dfrac{(k+1)((k+1)^2-1)}{3}$

so $P(k) \Rightarrow P(k+1)$

Since $P(1)$ is true and $P(k) \Rightarrow P(k+1)$ for $k \in \mathbb{Z}^+$ then by the principle of mathematical induction, the statement is true for all positive integers.

e Best proved by direct argument:

$n^3 - n = n(n^2 - 1) = (n-1)n(n+1)$
this is the product of three consecutive positive integers
(in the case $n = 1$, 0 is divisible by 3 so done)
Three consecutive positive integers always include
a multiple of 3, so the product is always divisible by 3

Exercise 1H

1

$8!-6!$	$6!(56-1) = 39600$
$9!+8!$	$8!(9+1) = 403200$
$7!-6!$	$6!(7-1) = 4320$
$6!+5!$	$5!(6+1) = 840$
$(n+1)!-n!$	$n!(n+1-1) = n(n!)$
$n!-(n-1)!$	$(n-1)!(n-1) = (n-1)(n-1)!$
$n!+(n-1)!$	$(n-1)!(n+1) = (n+1)(n-1)!$
$(n+1)!+n!$	$n!(n+1+1) = (n+2)n!$

2 **a** $\frac{8!}{4\times 6!} = \frac{8\times 7\times 6!}{4\times 6!} = 14$

b $\frac{4! \times 5!}{3! \times 6!} = \frac{(4)(3!)(5!)}{(3!)(6)(5!)} = \frac{4}{6} = \frac{2}{3}$

c $\frac{(10!)(8!)}{(11!)(6!)} = \frac{(10!)(8)(7)(6!)}{(11)(10!)(6!)} = \frac{56}{11}$

3 **a** $\frac{(n+1)!}{n! - (n+1)!} = \frac{(n+1)!}{n!(1-(n+1))} = -\frac{n+1}{n}$

b $\frac{n! + (n+1)!}{n!} = \frac{n!(1+n+1)}{n!} = n+2$

c $\frac{(n!)^2 - 1}{n! - 1} = \frac{(n!+1)(n!-1)}{n!-1} = 1+n!$

4 $\frac{(2n+2)!(n!)^2}{[(n+1)!]^2(2n)!} = \frac{(2n+2)(2n+1)}{(n+1)^2} = \frac{2(2n+1)}{n+1}$

5 ${}^nC_2 = \frac{n!}{(n-2)!2!} = 66$

$\Rightarrow \dfrac{n!}{(n-2)!} = n(n-1) = 132$

$\Rightarrow n^2 - n - 132 = 0$

$\Rightarrow (n-12)(n+11) = 0$

$n > 0$ so $n = 12$

6 $16(n-1)! = 5n! + (n+1)!$

$\Rightarrow 16 = 5n + (n+1)n$

$\Rightarrow n^2 + 6n - 16 = 0$

$\Rightarrow (n+8)(n-2) = 0$

$\Rightarrow n = 2 \quad (n > 0)$

7 **a** $13!$

b $4!(4! \times 3! \times 2! \times 4!) = 165888$

8 $26 \times 25 \times 24 \times 10 \times 9 = 1404000$

9 **a** ${}^{23}C_5 = 33649$

b Number of ways of choosing all boys = ${}^{13}C_5$

Number of ways of choosing all girls ${}^{10}C_5$

Number of ways of choosing at least one boy and at least one girl

$= {}^{23}C_5 - \left({}^{13}C_5 + {}^{10}C_5\right) = 32110$

10 **a** $6 \times 7^3 = 2058$

b $6 \times 6 \times 5 \times 4 = 720$

c Last digit must be 0, 4 or 8

$6 \times 7 \times 7 \times 3 = 882$

d Last digit must be 0

$6 \times 7 \times 7 \times 1 = 294$

11 ${}^{6}C_4 = 15$

12 There are ${}^{5}C_3$ ways to choose the drivers.

Then, there are 9 ways to choose passenger for small car.
This leaves 8 persons to choose 4 passenghers for second car and the rest go in the third car.

$\therefore \left({}^{5}C_3 \times 3!\right) \times 9 \times {}^{8}C_4 \times {}^{4}C_4 = 37800$

Exercise 1I

1 a $\left(1-\frac{x}{3}\right)^{11}=1+(11)\left(-\frac{x}{3}\right)+\frac{(11)(10)}{2!}\left(-\frac{x}{3}\right)^2+\frac{(11)(10)(9)}{3!}\left(-\frac{x}{3}\right)^3+\ldots$

$=1-\frac{11x}{3}+\frac{55x^2}{9}-\frac{55x^3}{9}+\ldots$

b $\left(1+\frac{x}{2}\right)^{7}=1+7\left(\frac{x}{2}\right)+\frac{(7)(6)}{2!}\left(\frac{x}{2}\right)^2+\frac{(7)(6)(5)}{3!}\left(\frac{x}{2}\right)^3+\ldots$

$=1+\frac{7x}{2}+\frac{21x^2}{4}+\frac{35x^3}{8}+\ldots$

c $\left(x+\frac{2}{x}\right)^{8}=x^8\left(1+\frac{2}{x^2}\right)^8$

$=x^8\left[1+8\left(\frac{2}{x^2}\right)+\frac{(8)(7)}{2!}\left(\frac{2}{x^2}\right)^2+\frac{(8)(7)(6)}{3!}\left(\frac{2}{x^2}\right)^3+\ldots\right]$

$=x^8+16x^6+112x^4+448x^2+\ldots$

2 a ${}^{10}C_4(a)^6(-2b)^4=3360a^6b^4$

b ${}^{11}C_2(a)^9\left(\frac{4}{a^2}\right)^2=880a^5$

c ${}^{8}C_3(x)^5\left(-\frac{2y}{x}\right)^3=-448x^2y^3$

3 General term is given by

${}^{12}C_r(x)^{12-r}\left(-\frac{2}{x^2}\right)^r=Nx^0$

Comparing powers of x

$12-r-2r=0$

$\Rightarrow r=4$

$\Rightarrow {}^{12}C_4(x)^8\left(-\frac{2}{x^2}\right)^4=7920$

4 $\left(2-\frac{x}{5}\right)^4=16\left(1-\frac{x}{10}\right)^4$

$=16\left({}^4C_0+{}^4C_1\left(-\frac{x}{10}\right)+{}^4C_2\left(-\frac{x}{10}\right)^2+{}^4C_3\left(-\frac{x}{10}\right)^3+{}^4C_4\left(-\frac{x}{10}\right)^4\right)$

$=16\left(1-\frac{2x}{5}+\frac{3x^2}{50}-\frac{x^3}{250}+\frac{x^4}{10000}\right)$

$=16-\frac{32x}{5}+\frac{24x^2}{25}-\frac{8x^3}{125}+\frac{x^4}{625}$

$\therefore (1.99)^4=\left(2-\frac{0.05}{5}\right)^4$

$16-\frac{32(0.05)}{5}+\frac{24(0.05)^2}{25}-\frac{8(0.05)^3}{125}+\frac{(0.05)^4}{625}$

$=15.68239$ to 5d.p.

5 General term is given by

$${}^6C_r\left(x^2\right)^{6-r}\left(-\frac{1}{x}\right)^r = Nx^6$$

Comparing powers of x

$12-2r-r=6$

$\Rightarrow r=2$

$${}^6C_2\left(x^2\right)^4\left(-\frac{1}{x}\right)^2 = 15x^6$$

6 a $$\left(x+\frac{y}{x}\right)^5 = x^5+5x^4\left(\frac{y}{x}\right)+10x^3\left(\frac{y}{x}\right)^2+10x^2\left(\frac{y}{x}\right)^3+5x\left(\frac{y}{x}\right)^4+\left(\frac{y}{x}\right)^5$$

$$= x^5+5x^3y+10xy^2+\frac{10y^3}{x}+\frac{5y^4}{x^3}+\frac{y^5}{x^5}$$

b $$(2x+y)\left(x^5+5x^3y+10xy^2+\frac{10y^3}{x}+\frac{5y^4}{x^3}+\frac{y^5}{x^5}\right)$$

Term in x^3y^2 is

$y\left(5x^3y\right)=5x^3y^2$ so 5

7 a $${}^{n+1}C_4=\frac{(n+1)!}{4!(n-3)!}$$

b $$2^3\times{}^nC_3=\frac{8\cdot n!}{(n-3)!3!}=\frac{4\cdot n!}{3\cdot(n-3)!}$$

c $$\frac{(n+1)!}{4!(n-3)!}=\frac{4\cdot n!}{3\cdot(n-3)!}$$

$$\Rightarrow n+1=\frac{4(4!)}{3}=32$$

$\Rightarrow n=31$

8 a $$\left(\sqrt{3}-\sqrt{2}\right)^5$$

$$=\left(\sqrt{3}\right)^5+5\left(\sqrt{3}\right)^4\left(-\sqrt{2}\right)+10\left(\sqrt{3}\right)^3\left(-\sqrt{2}\right)^2+10\left(\sqrt{3}\right)^2\left(-\sqrt{2}\right)^3$$
$$+5\left(\sqrt{3}\right)\left(-\sqrt{2}\right)^4+\left(-\sqrt{2}\right)^5$$

$$=9\sqrt{3}-45\sqrt{2}+60\sqrt{3}-60\sqrt{2}+20\sqrt{3}-4\sqrt{2}$$

$$=89\sqrt{3}-109\sqrt{2}$$

b $$\left(\sqrt{2}-\frac{1}{\sqrt{5}}\right)^4=\left(\sqrt{2}-\frac{\sqrt{5}}{5}\right)^4$$

$$=\left(\sqrt{2}\right)^4-4\left(\sqrt{2}\right)^3\left(\frac{\sqrt{5}}{5}\right)+6\left(\sqrt{2}\right)^2\left(\frac{\sqrt{5}}{5}\right)^2-4\left(\sqrt{2}\right)\left(\frac{\sqrt{5}}{5}\right)^3+\left(\frac{\sqrt{5}}{5}\right)^4$$

$$=4-\frac{8\sqrt{10}}{5}+\frac{12}{5}-\frac{4\sqrt{10}}{25}+\frac{1}{25}$$

$$=\frac{161}{25}-\frac{44}{25}\sqrt{10}$$

c $\left(1+\sqrt{5}\right)^7-\left(1-\sqrt{5}\right)^7$

$=2\left(7\sqrt{5}+35\left(\sqrt{5}\right)^3+21\left(\sqrt{5}\right)^5+\left(\sqrt{5}\right)^7\right)$

$=2\left(7\sqrt{5}+175\sqrt{5}+525\sqrt{5}+125\sqrt{5}\right)$

$=1664\sqrt{5}$

9 a ${}^nC_0-2\times{}^nC_1+4\times{}^nC_2-8\times{}^nC_3+\ldots+(-1)^r2^r\times{}^nC_r+\ldots+(-1)^n2^n\times{}^nC_n$

$=(1-2)^n=(-1)^n$

b ${}^nC_0+{}^nC_1+{}^nC_3+\ldots+{}^nC_n=(1+1)^n=2^n$

Exercise 1J

1 a $\dfrac{1}{1+x}=(1+x)^{-1}=1+(-1)x+\dfrac{(-1)(-2)}{2!}(x)^2+\dfrac{(-1)(-2)(-3)}{3!}(x)^3+\ldots$

$=1-x+x^2-x^3+\ldots$

b $\dfrac{1}{(1-2x)^2}=(1-2x)^{-2}$

$=1+(-2)(-2x)+\dfrac{(-2)(-3)}{2!}(-2x)^2+\dfrac{(-2)(-3)(-4)}{3!}(-2x)^3+\ldots$

$=1+4x+12x^2+32x^3+\ldots$

c Using the answer to part a and substituting $2x$ for x,

$\dfrac{2}{1+2x}=2(1+2x)^{-1}=2\left(1-2x+4x^2-8x^3+\ldots\right)$

$=2-4x+8x^2-16x^3+\ldots.$

d $\dfrac{2}{(1-x)^3}=2(1-x)^{-3}$

$2\left[1+(-3)(-x)+\dfrac{(-3)(-4)}{2!}(-x)^2+\dfrac{(-3)(-4)(-5)}{3!}(-x)^3+\ldots\right]$

$=2+6x+12x^2+20x^3+\ldots$

2 a $\sqrt{1+2x}=(1+2x)^{\frac{1}{2}}$

$=1+\dfrac{1}{2}(2x)+\dfrac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}(2x)^2+\dfrac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(2x)^3+\ldots$

$=1+x-\dfrac{1}{2}x^2+\dfrac{1}{2}x^3+\ldots$

b $(1+x)^{\frac{3}{2}}=1+\dfrac{3}{2}x+\dfrac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}{2!}x^2+\dfrac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{3!}x^3+\ldots$

$$1+\frac{3x}{2}+\frac{3x^2}{8}-\frac{x^3}{16}+\dots$$

c $(1-3x)^{-\frac{1}{2}}$

$$=1+\left(-\frac{1}{2}\right)(-3x)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}(-3x)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-3x)^3+\dots$$

$$=1+\frac{3x}{2}+\frac{27x^2}{8}+\frac{135}{16}x^3+\dots$$

d $2(1+x)^{\frac{1}{3}}$

$$=2\left[1+\frac{1}{3}x+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2}x^2+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}x^3+\dots\right]$$

$$=2\left[1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3+\dots\right]$$

$$=2+\frac{2x}{3}-\frac{2x^2}{9}+\frac{10x^3}{81}+\dots$$

3 $\sqrt{\dfrac{1-x}{1+x}}=(1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}$

$$=\left(1-\frac{1}{2}x+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}x^2-\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3+..\right)\left(1-\frac{1}{2}x+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}x^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}x^3+..\right)$$

$$=\left(1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}+\dots\right)\left(1-\frac{x}{2}+\frac{3x^2}{8}-\frac{5x^3}{16}+\dots\right)$$

$$=1-x+\frac{x^2}{2}-\frac{x^3}{2}+\dots$$

4 $\dfrac{x}{(1+x)^2}=x(1+x)^{-2}$

$$=x\left[1+(-2)x+\frac{(-2)(-3)}{2!}x^2+\frac{(-2)(-3)(-4)}{3!}x^3+\dots\right]$$

$$=x\left[1-2x+3x^2-4x^3+\dots\right]$$

$$=x-2x^2+3x^3-4x^4+\dots$$

5 $(2-3x)^{-3}=\dfrac{1}{8}\left(1-\dfrac{3}{2}x\right)^{-3}$

$$=\frac{1}{8}\left(1+(-3)\left(-\frac{3x}{2}\right)+\frac{(-3)(-4)}{2!}\left(-\frac{3x}{2}\right)^2+\frac{(-3)(-4)(-5)}{3!}\left(-\frac{3x}{2}\right)^3+\dots\right)$$

$$=\frac{1}{8}\left(1+\frac{9x}{2}+\frac{27x^2}{2}+\frac{135x^3}{4}+\dots\right)$$

$$=\frac{1}{8}+\frac{9x}{16}+\frac{27x^2}{16}+\frac{135x^3}{32}+\dots$$

6 a $\sqrt{1-4x} = (1-4x)^{\frac{1}{2}}$

$$= 1 + \frac{1}{2}(-4x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(-4x)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(-4x)^3 + \ldots$$
$$= 1 - 2x - 2x^2 - 4x^3 + \ldots$$

b $\sqrt{1-4\left(\frac{1}{100}\right)} = \sqrt{\left(\frac{96}{100}\right)} = \frac{4}{10}\sqrt{6} = \frac{2\sqrt{6}}{5}$

c $\sqrt{6} = \frac{5}{2}\sqrt{1-4\left(\frac{1}{100}\right)}$

$$= \frac{5}{2}\left(1 - 2\left(\frac{1}{100}\right) - 2\left(\frac{1}{100}\right)^2 - 4\left(\frac{1}{100}\right)^3 + \ldots\right)$$
$$= 2.44949$$

7 a $\frac{1}{\sqrt{1-2x}} = (1-2x)^{-\frac{1}{2}}$

$$= 1 + \left(-\frac{1}{2}\right)(-2x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-2x)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-2x)^3 \ldots.$$
$$= 1 + x + \frac{3x^2}{2} + \frac{5x^3}{2} + \ldots$$

b $\frac{(2+3x)^3}{\sqrt{1-2x}} = (2+3x)^3\left(1 + x + \frac{3x^2}{2} + \frac{5x^3}{2} + \ldots\right)$

Expanding

$$\left((2)^3 + 3(2)^2(3x) + 3(2)(3x)^2 + (3x)^3\right)\left(1 + x + \frac{3x^2}{2} + \frac{5x^3}{2} + \ldots\right)$$

$$\begin{aligned}
= 8 + 8x \;&+ 12x^2 + 20x^3 + \ldots \\
+36x &+ 36x^2 + 54x^3 + \ldots \\
&+54x^2 + 54x^3 + \ldots \\
&\qquad\quad + 27x^3 + \ldots. \\
\hline
= 8 + 44x &+ 102x^2 + 155x^3 + \ldots
\end{aligned}$$

Chapter review

1 $u_2 = u_1 r = 9 \Rightarrow u_1 = \dfrac{9}{r}$

$S_3 = u_1\left(1 + r + r^2\right) = 91$

$\Rightarrow \dfrac{9}{r}\left(1 + r + r^2\right) = 91$

$\Rightarrow 9 + 9r + 9r^2 = 91r$

$\Rightarrow 9r^2 - 82r + 9 = 0$

$\Rightarrow (9r - 1)(r - 9) = 0$

$\Rightarrow r = \dfrac{1}{9}$ or $r = 9$

Therefore there are two geometric sequences:

$r = \dfrac{1}{9} \Rightarrow u_4 = \dfrac{1}{9}$

$r = 9: \quad \Rightarrow u_4 = 729$

2 $u_1 = 1$

$1 + 2 + 3 + 4 + 5 + 7 + 8 + 9 + 11 + 13 + 15 + 16 + 17 + \ldots + 64$

$= \underbrace{(1 + 3 + 5 + 7 + \ldots + 63)}_{\substack{\text{arithmetic series} \\ \text{sum of first 32 odd numbers}}} + \underbrace{(2 + 4 + 8 + 16 + \ldots + 64)}_{\text{Finite geometric series,u=2, r=2, n=6}}$

$= \dfrac{32}{2}(1 + 63) + \dfrac{2(2^6 - 1)}{2 - 1}$

$= 1024 + 126$

$= 1150$

3 $b = a + d, c = a + 2d \Rightarrow a + d = 12 \Rightarrow a = 12 - d$

$\dfrac{c}{b} = \dfrac{a}{c} \Rightarrow \dfrac{a + 2d}{a + d} = \dfrac{a}{a + 2d}$

$\Rightarrow (a + 2d)^2 = a(a + d)$

Substituting for a

$\Rightarrow (12 - d + 2d)^2 = 12(12 - d)$

$\Rightarrow (12 + d)^2 = 144 - 12d$

$\Rightarrow 144 + 24d + d^2 - 144 + 12d = 0$

$\Rightarrow d^2 + 36d = d(d + 36) = 0$

$d \neq 0,\ d = -36$

$a = 48$

$b = 48 - 36 = 12$

$c = 48 - 72 = -24$

4 **a** $\dfrac{1}{1 + x} - \dfrac{1}{3\left(1 + \frac{2}{3}x\right)} = \dfrac{1}{1 + x} - \dfrac{1}{3 + 2x}$

$= \dfrac{3 + 2x - (1 + x)}{(1 + x)(3 + 2x)} = \dfrac{x + 2}{3 + 2x + 3x + 2x^2} = \dfrac{x + 2}{2x^2 + 5x + 3}$

b $\dfrac{x + 2}{2x^2 + 5x + 3} = (1 + x)^{-1} - \dfrac{1}{3}\left(1 + \dfrac{2}{3}x\right)^{-1}$

$$= \left(1 - x + x^2 - x^3 + \ldots\right) - \frac{1}{3}\left(1 - \frac{2}{3}x + \frac{4}{9}x^2 - \frac{8}{27}x^3 + \ldots\right)$$

$$= \frac{2}{3} - \frac{7}{9}x + \frac{23}{27}x^2 - \frac{73}{81}x^3 + \ldots$$

5 **a** $^nC_2 + n = \dfrac{n!}{(n-2)!2!} + n = \dfrac{1}{2}n(n-1) + n$

$$= \frac{1}{2}n(n-1+2) = \frac{1}{2}n(n+1) = \frac{1}{2}\frac{(n+1)!}{(n-1)!}$$

$$= \frac{(n+1)!}{2!(n-1)!} = {}^{n+1}C_2$$

b $^nC_2 \times {}^{n-2}C_{k-2} = \dfrac{n!}{2!(n-2)!} \times \dfrac{(n-2)!}{(n-k)!(k-2)!}$

$$= \frac{n!}{2!(n-k)!(k-2)!} = \frac{n!}{(n-k)!} \times \frac{1}{2!(k-2)!}$$

$$= \frac{n!k!}{(n-k)!k!} \times \frac{1}{2!(k-2)!}$$

$$= \frac{n!}{(n-k)!k!} \times \frac{k!}{2!(k-2)!}$$

$$= {}^nC_k \times {}^kC_2$$

6 $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_rx^r + \ldots + {}^nC_nx^n$

$\therefore {}^nC_0 + {}^nC_1 \times 3 + {}^nC_2 \times 3^2 + \ldots + {}^nC_r \times 3^r + \ldots + {}^nC_n \times 3^n$

$= (1+3)^n = 4^n = \left(2^2\right)^n = 2^{2n}$

7 Suppose there exist integers a and b such that $14a + 7b = 1$. Then, $2a + b = \dfrac{1}{7}$.

But the left-hand side is an integer whereas the right-hand side is not.
This is a contradiction. Therefore there are no such integers.

8 Suppose $x = 3$ and $5x - 7 = 13$. Then, $x = \dfrac{13+7}{5} = 4$. But $x = 3$, so this is a contradiction

9 **a** Take, for example, $a = 0$ and $b = 1$

b Take, for example, $n = 5$: $3^5 + 2 = 245 = 5(49)$ which is not prime

c Take, for example, $n = 1$: $\sqrt{2(1)-1} = \sqrt{1} = 1$ which is rational

d Take, for example, $n = 1$: $2^1 - 1 = 1$ and 1is not prime

10 $P(n)$: $(1 \times 1!) \times \left(2^2 \times 2!\right) \times \left(3^3 \times 3!\right) \times \ldots \times \left(n^n \times n!\right) = (n!)^{n+1}$

When $n = 1$

LHS= $1 \times 1! = 1$

RHS=$(1!)^{1+1} = 1^2 = 1$

LHS=RHS

$\therefore P(1)$ is true

Assume the statement $P(k)$ is true for some $k \in \mathbb{Z}^+$

i.e. $(1\times 1!)\times(2^2\times 2!)\times\ldots(k^k\times k!)=(k!)^{k+1}$

When $n=k+1$

LHS= $\underbrace{(1\times 1!)\times(2^2\times 2!)\times\ldots(k^k\times k!)}_{\text{use assumption}}\times\left((k+1)^{k+1}\times(k+1)!\right)$

$=(k!)^{k+1}\cdot(k+1)!(k+1)^{k+1}$

Regrouping

$=\left((k+1)k!\right)^{k+1}(k+1)!$

$=\left((k+1)!\right)^{k+1}(k+1)!$

$=\left((k+1)!\right)^{k+2}$

so $P(k)\Rightarrow P(k+1)$

Therefore, it has been shown that $P(1)$ is true and that if $P(k)$ is true for some $k \in \mathbb{Z}^+$ then so is $P(k+1)$. Therefore, the statement is true for all positive integers by the principle of mathematical induction

11 $P(n)$: $n^3+2n = 3A,\ A \in \mathbb{Z}$

When $n=1$

$1^3+2(1)=3$

The statement $P(1)$ is true

Assume that $P(k)$ is true for some $k \in \mathbb{Z}^+$

$k^3+2k=3m$ for some $m \in \mathbb{Z}^+$

$\Rightarrow k^3=3m-2k$

When $n=k+1$

LHS=$(k+1)^3+2(k+1)$

$=k^3+3k^2+3k+1+2k+2$

$=3m-2k+3k^2+5k+3$

$=3(m+k^2+k+1)$

$\therefore P(k)\Rightarrow P(k+1)$

Therefore, it has been shown that $P(1)$ is true and that if $P(k)$ is true for some $k \in \mathbb{Z}^+$ then so is $P(k+1)$. Therefore, the statement is true for all positive integers by the principle of mathematical induction.

12 a $P(n)$: $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

When $n=1$

LHS= $\sum_{r=1}^{1} r = 1$

RHS= $= \frac{1(1+1)}{2}=1$

$P(1)$ is true

Assume that $P(k)$ is true for some $k \in \mathbb{Z}^+$

i.e. $\sum_{r=1}^{k} r = \frac{k(k+1)}{2}$

When $n = k+1$,

$\sum_{r=1}^{k+1} r = \sum_{r=1}^{k} r + (k+1) = \frac{k(k+1)}{2} + \frac{(k+1)}{1}$

$= \frac{(k+1)(k+2)}{2}$

so $P(k) \Rightarrow P(k+1)$

Therefore, it has been shown that $P(1)$ is true and that if $P(k)$ is true for some $k \in \mathbb{Z}^+$ then so is $P(k+1)$. Therefore, the statement is true for all positive integers by the principle of mathematical induction

b $P(n)$: $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

When $n = 1$

LHS= $\sum_{r=1}^{1} r^2 = 1$

RHS= $= \frac{1(1+1)(2+1)}{6} = 1$

$P(1)$ is true

Assume that $P(k)$ is true for some $k \in \mathbb{Z}^+$

i.e. $\sum_{r=1}^{k} r^2 = \frac{k(k+1)(2k+1)}{6}$

When $n = k+1$,

$\sum_{r=1}^{k+1} r^2 = \sum_{r=1}^{k} r^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

$= \frac{k+1}{6}\left[k(2k+1) + 6(k+1)\right] = \frac{k+1}{6}(2k^2 + 7k + 6)$

$= \frac{k+1}{6}(2k+3)(k+2)$

$= \frac{(k+1)(k+2)(2(k+1)+1)}{6}$

so $P(k) \Rightarrow P(k+1)$

Therefore, it has been shown that $P(1)$ is true and that if $P(k)$ is true for some $k \in \mathbb{Z}^+$ then so is $P(k+1)$. Therefore, the statement is true for all positive integers by the principle of mathematical induction

c $P(n)$: $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$

When $n = 1$

LHS= $\sum_{r=1}^{1} r^3 = 1$

RHS= $= \frac{1^2(1+1)^2}{4} = 1$

$P(1)$ is true

Assume that $P(k)$ is true for some $k \in \mathbb{Z}^+$

i.e. $\sum_{r=1}^{k} r^3 = \frac{k^2(k+1)^2}{4}$

When $n = k+1$,

$$\sum_{r=1}^{k+1} r^3 = \sum_{r=1}^{k} r^3 + (k+1)^3$$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= \frac{(k+1)^2}{4}\left[k^2 + 4(k+1)\right]$$

$$= \frac{(k+1)^2(k^2+4k+4)}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

so $P(k) \Rightarrow P(k+1)$

Therefore, it has been shown that $P(1)$ is true and that if $P(k)$ is true for some $k \in \mathbb{Z}^+$ then so is $P(k+1)$. Therefore, the statement is true for all positive integers by the principle of mathematical induction

$$\therefore \sum_{r=1}^{n} r(r+1)(r+2) = \sum_{r=1}^{n}(r^3 + 3r^2 + 2r)$$

$$= \sum_{r=1}^{n} r^3 + 3\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r$$

$$= \frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{2} + n(n+1)$$

$$= \frac{n(n+1)}{4}\left[n(n+1) + 2(2n+1) + 4\right]$$

$$= \frac{n(n+1)}{4}\left[n^2 + 5n + 6\right]$$

$$= \frac{n(n+1)(n+2)(n+3)}{4}$$

13 a 'harmonics' consists of 9 different letters, so there are $9!$ arrangements.

b 5 digit numbers:

4 ways of choosing first digit (bigger than 3)

Each of the next three digits can be chosen in 7 ways

The last digit can be 0 or 5

These numbers include 30000 which is not wanted

In all there are $4 \times 7^3 \times 2 - 1$ five digit numbers

6 digit numbers:

6 ways of choosing first digit

7 ways of choosing each of the next four digits

2 ways of choosing last digit Divisible by 5 $\Rightarrow$ final digit is 0 or 5

In all there are $6 \times 7^4 \times 2$ six digit numbers

7 digit numbers:

6 ways of choosing first digit

7 ways of choosing each of the next five digits

2 ways of choosing last digit

In all there are $6\times7^5\times2$ six digit numbers

$$\text{Answer } = \left(4\times7^3\times2-1\right)+\left(6\times7^4\times2\right)+\left(6\times7^5\times2\right)$$
$$=2743+28812+201684$$
$$=233239$$

c The only possibilities would be to have 3 women and 2 men or 4 women and 1 man

$${}^4C_3\times{}^7C_2+{}^4C_4\times{}^7C_1=4\times21+1\times7=91$$

14 a $a^2-b^2=(a+b)(a-b)=(2x)(2y)=4xy$

b $a^3=(x+y)^3=x^3+3x^2y+3xy^2+y^3$

$b^3=(x-y)^3=x^3-3x^2y+3xy^2-y^3$

c $\therefore a^3-b^3=\left(x^3+3x^2y+3xy^2+y^3\right)-\left(x^3-3x^2y+3xy^2-y^3\right)$

$=2\left(3x^2y+y^3\right)$
$=(2y)\left(3x^2+y^2\right)$
$=(a-b)\left(3x^2+y^2\right)$
But,
$a^2+ab+b^2=3x^2+y^2$
So,
$a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$

d $a^4=(x+y)^4=x^4+4x^3y+6x^2y^2+4xy^3+y^4$

$b^4=(x-y)^4=x^4-4x^3y+6x^2y^2-4xy^3+y^4$
$\therefore a^4-b^4=\left(x^4+4x^3y+6x^2y^2+4xy^3+y^4\right)-\left(x^4-4x^3y+6x^2y^2-4xy^3+y^4\right)$
$=8x^3y+8xy^3$
$=(2y)\left(4x^3+4xy^2\right)$
$=(a-b)\left(a^3+a^2b+ab^2+b^3\right)$

e Conjecture: $a^n-b^n=(a-b)\left(a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots+ab^{n-2}+b^{n-1}\right)$ □

f $P(n): a^n-b^n=(a-b)\left(a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots+ab^{n-2}+b^{n-1}\right)$

When n=2
LHS=a^2-b^2
RHS $=(a-b)(a+b)=a^2+ab-ab+b^2=$ LHS
$P(2)$ is true
Assume that $P(k)$ is true for some $k\in\mathbb{Z}^+$
i.e. $a^k-b^k=(a-b)\left(a^{k-1}+a^{k-2}b+a^{k-3}b^2+\ldots+ab^{k-1}\right)$
$\Rightarrow a^k=(a-b)\left(a^{k-1}+a^{k-2}b+a^{k-3}b^2+\ldots+ab^{k-1}\right)+b^k$

When $n = k+1$

$$a^{k+1} - b^{k+1} = a\left(a^k\right) - b^{k+1}$$
$$= a\left[(a-b)\left(a^{k-1} + a^{k-2}b + a^{k-3}b^2 + \ldots + ab^{k-1}\right) + b^k\right] - b^{k+1}$$
$$= a(a-b)\left(a^{k-1} + a^{k-2}b + a^{k-3}b^2 + \ldots + ab^{k-1}\right) + ab^k - b^{k+1}$$
$$= a(a-b)\left(a^{k-1} + a^{k-2}b + a^{k-3}b^2 + \ldots + ab^{k-1}\right) + b^k(a-b)$$
$$= (a-b)\left[a\left(a^{k-1} + a^{k-2}b + a^{k-3}b^2 + \ldots + ab^{k-1}\right) + b^k\right]$$
$$= (a-b)\left(a^k + a^{k-1}b + a^{k-2}b^2 + \ldots + ab^{k-1} + b^k\right)$$

so $P(k) \Rightarrow P(k+1)$

Therefore, it has been shown that $P(2)$ is true and that if $P(k)$ is true for some $k \in \mathbb{Z}^+$, $k \geq 2$ then $P(k+1)$is also true. Therefore, the statement is true for all positive integers greater than 2, by the principle of mathematical induction

15 The difference between the coefficients must be the same

$$\therefore {}^nC_r - {}^nC_{r-1} = {}^nC_{r+1} - {}^nC_r$$
$$\Rightarrow \frac{n!}{r!(n-r)!} - \frac{n!}{(r-1)!(n-r+1)!} = \frac{n!}{(r+1)!(n-r-1)!} - \frac{n!}{r!(n-r)!}$$

Multiplying by $\dfrac{(r+1)!(n-r+1)!}{n!}$,

$$(r+1)(n-r+1) - (r+1)(r) = (n-r+1)(n-r) - (r+1)(n-r+1)$$
$$\Rightarrow 2(r+1)(n-r+1) - r(r+1) - (n-r+1)(n-r) = 0$$
$$\Rightarrow (n-r+1)(3r+2-n) - r^2 - r = 0$$

which after expanding and simplifing gives

$$n^2 + 4r^2 - 2 - n(4r+1) = 0$$

16 $\dfrac{2+x-7x^2}{(1-2x)\left(1-x^2\right)} = \dfrac{A}{1-2x} + \dfrac{B}{1+x} + \dfrac{C}{1-x}$

$$\therefore 2 + x - 7x^2 = A(1+x)(1-x) + B(1-2x)(1-x) + C(1-2x)(1+x)$$

Set $x = 1$: $\quad -4 = -2C \Rightarrow C = 2$

Set $x = -1$: $\quad -6 = 6B \Rightarrow B = -1$

Compare constants: $\; 2 = A + B + C \Rightarrow A = 2 - B - C = 1$

$$\therefore \frac{2+x-7x^2}{(1-2x)(1-x^2)} = \frac{1}{1-2x} - \frac{1}{1+x} + \frac{2}{1-x}$$
$$= (1-2x)^{-1} - (1+x)^{-1} + 2(1-x)^{-1}$$
$$= \left(1 + 2x + 4x^2 + 8x^3 + \ldots\right) - \left(1 - x + x^2 - x^3 + \ldots\right) + 2\left(1 + x + x^2 + x^3 + \ldots\right)$$
$$= 2 + 5x + 5x^2 + 11x^3 + \ldots$$

Exam-style questions

17 Require ($3\times$ coefficient of term in x^5) $+\left(1\times\right.$coefficient of term in $\left.x^4\right)$

$$3\times\binom{8}{5}4^3(-2x)^5 + 1\times\binom{8}{4}4^4(-2x)^4$$ (3 marks)

$$= 3\times(-114688) + 1\times 286720$$

$= -57344$ (1 mark)

18 $\binom{n}{2}\left(1^{n-2}\right)(3x)^2 = 495x^2$ (2 marks)

$\frac{9n(n-1)}{2} = 495$

$n(n-1) = 110$

$n^2 - n - 110 = 0$ (1 mark)

$(n-11)(n+10) = 0$ (2 marks)

So $n = 11$ or $n = -10$ (1 mark)

19 First part is geometric sum, $a = 1$, $r = 1.6$, $n = 16$ (1 mark)
Second part is arithmetic sum, $a = 0$, $d = -12$, $n = 16$ (1 mark)
Third part is $16 \times 1 = 16$ (1 mark)

Geometric sum: $S_{16} = \frac{1.6^{16} - 1}{1.6 - 1} = 3072.791$ (1 mark)

Arithmetic sum: $S_{16} = \frac{16}{2}\left(2 \times 0 + 15 \times (-12)\right) = -1440$ (1 mark)

So $\sum_{n=0}^{n=15}\left(1.6^n - 12n + 1\right)$

$= 3072.791 - 1440 + 16$

$= 1648.8$ (1 mark)

20 $\binom{n-1}{k} + \binom{n-1}{k-1}$

$= \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!}$ (3 marks)

$= \frac{(n-k)(n-1)! + k(n-1)!}{k!(n-k)!}$ (1 mark)

$= \frac{n(n-1)! - k(n-1)! + k(n-1)!}{k!(n-k)!}$ (1 mark)

$= \frac{n(n-1)!}{k!(n-k)!}$ (1 mark)

$= \frac{n!}{k!(n-k)!}$

$= \binom{n}{k}$

21 Consider multiples of 7:

504 is the first multiple and 1400 is the final multiple

$1400 = 504 + 7(n-1)$ (1 mark)

$\Rightarrow n = 129$ (1 mark)

So the sum of the multiples of 7 is $S_{129} = \frac{129}{2}\left(2\times 504 + 7\times(129-1)\right) = 122\,808$ (2 marks)

Sum of the integers from 500 to 1400 (inclusive) is

$S_{901} = \frac{901}{2}\left(2\times 500 + 1\times(901-1)\right) = 855\,950$ (2 marks)

Therefore require $855\,950 - 122\,808 = 733\,142$ (1 mark)

22 Suppose $n^3 + 3$ is odd. Assume, for a contradiction, that n is also odd. (1 mark)

Then we can write $n = 2p + 1$ for $p \in \mathbb{Z}^+$ and $n^3 + 3 = 2q + 1$ for $q \in \mathbb{Z}^+$. (1 mark)

So $n^3 + 3 = 2q + 1$

$(2p+1)^3 + 3 = 2q + 1$ (1 mark)

$8p^3 + 12p^2 + 6p + 1 + 3 = 2q + 1$ (1 mark)

$8p^3 + 12p^2 + 6p + 3 = 2q$

So $q = 4p^3 + 6p^2 + 3p + \frac{3}{2}$ (1 mark)

Since p is an integer, $4p^3 + 6p^2 + 3p$ is also an integer.

Since $\frac{3}{2}$ is a non-integer, then $4p^3 + 6p^2 + 3p + \frac{3}{2}$ is also a non-integer. (1 mark)

This is a contradiction, since q was assumed to be an integer. (1 mark)

Therefore, the initial assumption is false, and n must be even.

23 Case $n = 1$:

$5^{2(1)-1} + 1 = 5 + 1 = 6 = 1\times 6$ (1 mark)

Therefore true for $n = 1$

Case $n = k$:

Assume the statement is true for some $k \in \mathbb{Z}$, $k \geq 0$ (1 mark)

Then $5^{2k-1} + 1 = 6s$ for some positive integer s

Now $5^{2(k+1)-1} + 1$ (1 mark)

$= 5^{2k+2-1} + 1$

$= 5^2 \times 5^{2k-1} + 1$ (1 mark)

$= 5^2 \times (6s - 1) + 1$ (1 mark)

$= 25(6s-1)+1$

$= 25 \times 6s - 24$ (1 mark)

$= 6(25s-4)$

Which is a multiple of 6 (1 mark)

So the statement is true for $n=1$, and when assumed true for $n=k$,

is true for $n=k+1$.

Therefore the statement is true for all $n \in \mathbb{Z}^+$. (1 mark)

24 a $\sqrt[3]{1-x} = (1-x)^{\frac{1}{3}}$ (1 mark)

$= 1 - \frac{x}{3} + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)(-x)^2}{2!} + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)(-x)^3}{3!} + \cdots$ (2 marks)

$= 1 - \frac{x}{3} - \frac{x^2}{9} - \frac{5x^3}{81}$ (1 mark)

b When $x = \frac{1}{64}$, (1 mark)

$\sqrt[3]{1-x} = \sqrt[3]{1-\frac{1}{64}} = \sqrt[3]{\frac{63}{64}} = \frac{\sqrt[3]{63}}{4}$ (1 mark)

Therefore, when $x = \frac{1}{64}$, then

$\sqrt[3]{63} \approx 4\left[1 - \frac{x}{3} - \frac{x^2}{9} - \frac{5x^3}{81}\right]$ (1 mark)

$= 4\left[1 - \frac{\left(\frac{1}{64}\right)}{3} - \frac{\left(\frac{1}{64}\right)^2}{9} - \frac{5\left(\frac{1}{64}\right)^3}{81}\right]$ (1 mark)

$= 4 - \frac{4}{192} - \frac{4}{36\,864} - \frac{20}{21\,233\,664}$

$= 3.979057$ (1 mark)

25 a $9! = 362\,880$ (1 mark)

b $2 \times 8! = 80\,640$ (2 marks)

c $9! - 2 \times 8! = 282\,240$ (2 marks)

d We require:

(no. of ways in total) – (no. of ways with one woman separating men)

– (no. ways with men together) (1 mark)

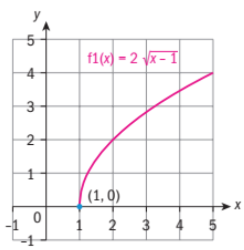
$= 9! - 2 \times 7 \times 7! - 2 \times 8!$ (1 mark)

$= 211680$ (1 mark)

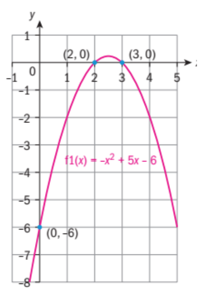
2 Representing relationships: functions

Skills check

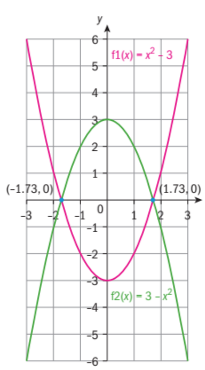
1 a



b

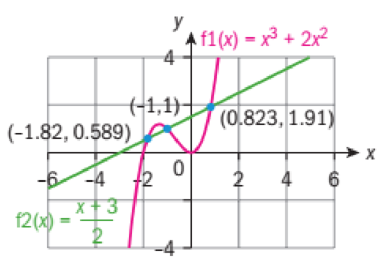


2 a



The graphs intersect at (-1.73,0) and (1.73,0), each to 3sf.

b



The graphs intersect at (-1,1), (-1.82,0.589) to 3sf, and (0.823,1.91) to 3 sf.

3 a $y = x^2 - 2x + 3$
$= (x^2 - 2x + 1) + 3 - 1$
$= (x - 1)^2 + 2$

b $y = -x^2 - 6x + 1$
$= -(x^2 + 6x + 9) + 1 + 9$
$= -(x + 3)^2 + 10$

c $y = 3x^2 + 6x + 1$
$= 3(x^2 + 2x + 1) + 1 - 3$
$= 3(x + 1)^2 - 2$

Exercise 2A

1 a Yes

$D_f = \{1, 2, 3, 4\}$
$R_f = \{0, 2, 3, 4\}$

b Yes

$D_f = \{-2, -1, 0, 1\}$
$R_f = \{1\}$

c No, this is not a function because it is not well-defined: 2 is mapped to multiple values

d No, this is not a function because it is not well-defined: π is mapped to both π and π^π

e Yes

$D_f = \{1, 2, 3, 4, 5\}$
$R_f = \{2, 4, 10\}$

f No, this is not a function because it is not well-defined: -5 is mapped to both 0 and 1

g No, this is not a function, since it is does not act on the entire domain: 5 has no image

h No, this is not a function, because it is not well-defined: 2 is mapped to both 8 and 15

2 a No, because the graph does not pass the vertical line test

b Yes

$D_f = $ □
$R_f = \{2\}$

c No, because the graph does not pass the vertical line test

d Yes

$D_f = \{x \in \mathbb{R} \mid 1 < x < 6\}$
$R_f = \{y \in \mathbb{R} \mid 1 \le y \le 7\}$

e Yes

$D_f = \{-4, -3, -2, -1, 1, 2, 3, 4\}$
$R_f = \{-3, -2, -1, 0, 1, 2, 4\}$

f Yes

$D_f = \{x \in \mathbb{R} \mid -4 < x < 3\}$
$R_f = \{y \in \mathbb{R} \mid -2 \le y \le 1\}$

g Yes

$D_f = R_f = \mathbb{R}$

Exercise 2B

1 a i $y = x^2 + 6x + 8 = (x+3)^2 + 8 - 9 = (x+3)^2 - 1$

So the axis of symmetry is $x = -3$

ii $(-3, -1)$

iii Concavity: up, $D_f = \mathbb{R}$, $R_f = \{y \in \mathbb{R} \mid y \ge -1\}$

b i $y = 10 + 3x - x^2 = -(x^2 - 3x - 10) = -\left(\left(x - \frac{3}{2}\right)^2 - \frac{9}{4} - 10\right) = \frac{49}{4} - \left(x - \frac{3}{2}\right)^2$

So the axis of symmetry is $x = \frac{3}{2}$

ii $\left(\frac{3}{2}, \frac{49}{4}\right)$

iii Concave down, $D_f = \mathbb{R}$, $R_f = \left\{y \in \mathbb{R} \mid y \le \frac{49}{4}\right\}$

c i $y = 3\left(x^2 - 4x - \frac{5}{3}\right) = 3\left((x-2)^2 - \frac{17}{3}\right) = 3(x-2)^2 - 17$

So the axis of symmetry is $x = 2$

ii $(2, -17)$

iii Concave up, $D_f = \mathbb{R}$, $R_f = \{y \in \mathbb{R} \mid y \ge -17\}$

d i $y = -2\left(x^2 + 2x - \frac{7}{2}\right) = -2\left((x+1)^2 - \frac{9}{2}\right) = 9 - 2(x+1)^2$

So the axis of symmetry is $x = -1$

ii $(-1, 9)$

iii Concave down, $D_f = \mathbb{R}$, $R_f = \{y \in \mathbb{R} \mid y \le 9\}$

2 **a** Vertex is $(2, -16) \Rightarrow y = a(x-2)^2 - 16$

$-12 = a(0-2)^2 - 16 \Rightarrow a = 1$
$\therefore y = (x-2)^2 - 16$

b $x-$intercepts are $x = -3$, $x = 1$ so the quadratic must be of the form

$y = C(x+3)(x-1) = C(x^2 + 2x - 3)$
At $x = 0$, $3 = -3C$ so $C = -1$
$\therefore y = 3 - 2x - x^2$

c $x-$intercepts are $x = 1$, $x = 5$ so the quadratic must be of the form

$y = C(x-1)(x-5) = C(x^2 - 6x + 5)$
At $x = 4$, $-12 = C(-3) \Rightarrow C = 4$
$\therefore y = 4(x^2 - 6x + 5) = 4x^2 - 24x + 20$

d Vertex is $(2, -6) \Rightarrow y = a(x-2)^2 - 6$

$6 = a(4-2)^2 - 6 \Rightarrow a = 3$
$y = 3(x-2)^2 - 6$

e $x-$intercepts are $x = -5$, $x = 2$ so the quadratic must be of the form

$y = C(x+5)(x-2) = C(x^2 + 3x - 10)$

At $x = 1$, $3 = C(-6) \Rightarrow C = -\frac{1}{2}$

$\therefore y = -\frac{1}{2}(x^2 + 3x - 10) = 5 - \frac{3}{2}x - \frac{1}{2}x^2$

f Vertex is $(-10, 60) \Rightarrow y = a(x+10)^2 + 60$

$45 = a(-5+10)^2 + 60 \Rightarrow a = -\frac{3}{5}$

$y = \frac{-3}{5}(x+10)^2 + 60$

Exercise 2C

1 $4 - 2x \ne 0$ therefore $x \ne 2$ and $D_f = \{x \in \mathbb{R} \mid x \ne 2\}$

$y = \frac{3}{4-2x} \Rightarrow 4y - 2xy = 3 \Rightarrow x = \frac{4y-3}{2y}$
$\therefore y \ne 0$ and $R_f = \{y \in \mathbb{R} \mid y \ne 0\}$

Asymptotes: $x = 2$ and $y = 0$

2 $3 - 6x \neq 0 \Rightarrow x \neq \frac{1}{2}$ so $D_f = \left\{x \in \mathbb{R} \mid x \neq \frac{1}{2}\right\}$

$y = \frac{1}{6x - 3} \Rightarrow 6xy - 3y = 1 \Rightarrow x = \frac{1 + 3y}{6y}$

$\therefore y \neq 0$ and $R_f = \{y \in \mathbb{R} \mid y \neq 0\}$

Asymptotes: $x = \frac{1}{2}$ and $y = 0$

3 $2 - 4x \neq 0 \Rightarrow x \neq \frac{1}{2}$ so $D_f = \left\{x \in \mathbb{R} \mid x \neq \frac{1}{2}\right\}$

$y = \frac{x}{2 - 4x} \Rightarrow 2y - 4xy = x \Rightarrow x = \frac{2y}{1 + 4y}$

$\therefore y \neq -\frac{1}{4}$ and $R_f = \left\{y \in \mathbb{R} \mid y \neq -\frac{1}{4}\right\}$

Asymptotes: $x = \frac{1}{2}$ and $y = -\frac{1}{4}$

4 $1 - x \neq 0 \Rightarrow x \neq 1$ so $D_f = \{x \in \mathbb{R} \mid x \neq 1\}$

$y = \frac{1 + x}{1 - x} \Rightarrow y - yx = 1 + x \Rightarrow x = \frac{y - 1}{y + 1}$

$\therefore y \neq -1$ and $R_f = \{y \in \mathbb{R} \mid y \neq -1\}$

Asymptotes: $x = 1$ and $y = -1$

5 $1 + 2x \neq 0 \Rightarrow x \neq -\frac{1}{2}$ so $D_f = \left\{x \in \mathbb{R} \mid x \neq -\frac{1}{2}\right\}$

$y = \frac{1 - 2x}{1 + 2x} \Rightarrow y + 2xy = 1 - 2x \Rightarrow x = \frac{1 - y}{2(1 + y)}$

$\therefore y \neq -1$ and $R_f = \{y \in \mathbb{R} \mid y \neq -1\}$

Asymptotes: $x = -\frac{1}{2}$ and $y = -1$

6 $2 - 3x \neq 0 \Rightarrow x \neq \frac{2}{3}$ so $D_f = \left\{x \in \mathbb{R} \mid x \neq \frac{2}{3}\right\}$

$y = -\frac{2x - 3}{2 - 3x} = \frac{3 - 2x}{2 - 3x} \Rightarrow 2y - 3xy = 3 - 2x \Rightarrow x = \frac{2y - 3}{3y - 2}$

$\therefore y \neq \frac{2}{3}$ and $R_f = \left\{y \in \mathbb{R} \mid y \neq \frac{2}{3}\right\}$

Asymptotes: $x = \frac{2}{3}$ and $y = \frac{2}{3}$

Exercise 2D

1 a $y = \sqrt{x - 2}$

$x - 2 \geq 0$ so $D_f = \{x \in \mathbb{R} \mid x \geq 2\}$

$y \geq 0$ so $R_f = \{y \in \mathbb{R} \mid y \geq 0\}$

b $y = \sqrt{3x - 2}$

$3x - 2 \geq 0 \Rightarrow x \geq \dfrac{2}{3}$ and $D_f = \left\{x \in \mathbb{R} \mid x \geq \dfrac{2}{3}\right\}$

$y \geq 0 \Rightarrow R_f = \{y \in \mathbb{R} \mid y \geq 0\}$

c $y = 1 + \sqrt{2 - 4x}$

$2 - 4x \geq 0 \Rightarrow x \leq \dfrac{1}{2}$ and $D_f = \left\{x \in \mathbb{R} \mid x \leq \dfrac{1}{2}\right\}$

$y \geq 1$ so $R_f = \{y \in \mathbb{R} \mid y \geq 1\}$

d $y = 3 - \sqrt{2x + 1}$

$2x + 1 \geq 0 \Rightarrow x \geq -\dfrac{1}{2}$ and $D_f = \left\{x \in \mathbb{R} \mid x \geq -\dfrac{1}{2}\right\}$

$y \leq 3$ so $R_f = \{y \in \mathbb{R} \mid y \leq 3\}$

e $y = -2\sqrt{x - 1}$

$x - 1 \geq 0 \Rightarrow x \geq 1$ and $D_f = \{x \in \mathbb{R} \mid x \geq 1\}$

$y \leq 0$ so $R_f = \{y \in \mathbb{R} \mid y \leq 0\}$

f $y = 1 - 3\sqrt{2 - x}$

$2 - x \geq 0 \Rightarrow x \leq 2$ and $D_f = \{x \in \mathbb{R} \mid x \leq 2\}$

$y \leq 1 \Rightarrow R_f = \{y \in \mathbb{R} \mid y \leq 1\}$

Exercise 2E

1 $y = \dfrac{4}{x^2 - 3x} = \dfrac{4}{x(x - 3)}$

$x(x - 3) \neq 0 \Rightarrow x \neq 0$ and $x \neq 3$

$\therefore D_f = \{x \in \mathbb{R} \mid x \neq 0, x \neq 3\}$

$R_f = \{y \in \mathbb{R} \mid y \neq 0\}$

Asymptotes: $x = 0,\ x = 3,\ y = 0$

2 $y = \dfrac{1}{x^2 - 9} = \dfrac{1}{(x + 3)(x - 3)}$

$(x + 3)(x - 3) \neq 0 \Rightarrow x \neq \pm 3$

$\therefore D_f = \{x \in \mathbb{R} \mid x \neq \pm 3\}$

$R_f = \left\{y \in \mathbb{R} \mid y > 0 \text{ or } y \leq -\dfrac{1}{9}\right\}$

Asymptotes: $x = -3,\ x = 3,\ y = 0$

3 $y = -\dfrac{1}{x^2 + 2x - 3} = -\dfrac{1}{(x + 3)(x - 1)}$

$(x+3)(x-1) \neq 0 \Rightarrow x \neq 1,\ x \neq -3$

$\therefore D_f = \{x \in \mathbb{R} \mid x \neq 1,\ x \neq -3\}$

$R_f = \left\{y \in \mathbb{R} \mid y < 0 \text{ or } y \geq \frac{1}{4}\right\}$

Asymptotes: $x = -3,\ x = 1,\ y = 0$

4 $y = \dfrac{2}{(x+2)^2}$

$(x+2)^2 \neq 0 \Rightarrow x \neq -2$

$\therefore D_f = \{x \in \mathbb{R} \mid x \neq -2\}$

$R_f = \{y \in \mathbb{R} \mid y > 0\}$

Asymptotes: $x = -2,\ y = 0$

5 $y = -\dfrac{1}{2x^2+9x-18} = -\dfrac{1}{(2x-3)(x+6)}$

$\therefore D_f = \left\{x \in \mathbb{R} \mid x \neq \frac{3}{2},\ x \neq -6\right\}$

$R_f = \left\{y \in \mathbb{R} \mid y < 0 \text{ or } y \geq \frac{8}{225}\right\}$

Asymptotes: $x = -6,\ x = \frac{3}{2},\ y = 0$

6 $D_f = \{x \in \mathbb{R} \mid x > -2\}$

$R_f = \{y \in \mathbb{R} \mid y > 0\}$

Asymptotes: $x = -2,\ y = 0$

7 $y = \dfrac{1}{\sqrt{2x^2-3x-2}} = \dfrac{1}{\sqrt{(x-2)(2x+1)}}$

$\therefore D_f = \left\{x \in \mathbb{R} \mid x \neq 2, x \neq \frac{-1}{2}\right\}$

$R_f = \{y \in \mathbb{R} \mid y > 0\}$

Asymptotes: $y = 0, x = -\frac{1}{2}$ and $x = 2$

8 $y = -\dfrac{2}{\sqrt{4x^2-25}} = -\dfrac{2}{\sqrt{(2x+5)(2x-5)}}$

$\therefore D_f = \left\{x \in \mathbb{R} \mid x \leq -\frac{5}{2} \text{ or } x \geq \frac{5}{2}\right\}$

$R_f = \{y \in \mathbb{R} \mid y < 0\}$

Asymptotes: $x = -\frac{5}{2},\ x = \frac{5}{2},\ y < 0$

Exercise 2F

1 $\dfrac{1}{x^2+5x+6}=\dfrac{1}{(x+3)(x+2)}=\dfrac{A}{x+3}+\dfrac{B}{x+2}$

$\Rightarrow 1=A(x+2)+B(x+3)$
Set $x=-2$: $1=B\Rightarrow B=1$
Set $x=-3$: $1=-A\Rightarrow A=-1$
$\therefore \dfrac{1}{x^2+5x+6}=\dfrac{1}{x+2}-\dfrac{1}{x+3}$

2 $\dfrac{4-x}{x^2+x-2}=\dfrac{4-x}{(x+2)(x-1)}=\dfrac{A}{x+2}+\dfrac{B}{x-1}$

$\Rightarrow 4-x=A(x-1)+B(x+2)$
Set $x=1$: $3=3B\Rightarrow B=1$
Set $x=-2$: $6=-3A\Rightarrow A=-2$
$\therefore \dfrac{4-x}{x^2+x-2}=\dfrac{1}{x-1}-\dfrac{2}{x+2}$

3 $\dfrac{4x-9}{x^2-3x}=\dfrac{4x-9}{x(x-3)}=\dfrac{A}{x}+\dfrac{B}{x-3}$

$\Rightarrow 4x-9=A(x-3)+Bx$
Set $x=0$: $-9=-3A\Rightarrow A=3$
Set $x=3$: $3=3B\Rightarrow B=1$
$\therefore \dfrac{4x-9}{x^2-3x}=\dfrac{3}{x}+\dfrac{1}{x-3}$

4 $\dfrac{x}{x^2-1}=\dfrac{x}{(x+1)(x-1)}=\dfrac{A}{x+1}+\dfrac{B}{x-1}$

$\Rightarrow x=A(x-1)+B(x+1)$

Set $x=1$: $1=2B\Rightarrow B=\dfrac{1}{2}$

Set $x=-1$: $-1=-2A\Rightarrow A=\dfrac{1}{2}$

$\therefore \dfrac{x}{x^2-1}=\dfrac{1}{2}\left(\dfrac{1}{x+1}+\dfrac{1}{x-1}\right)$

5 $\dfrac{5}{-x^2-x+6}=-\dfrac{5}{x^2+x-6}=-\dfrac{5}{(x+3)(x-2)}=\dfrac{A}{x+3}+\dfrac{B}{x-2}$

$\Rightarrow -5=A(x-2)+B(x+3)$
Set $x=2$: $-5=5B\Rightarrow B=-1$
Set $x=-3$: $-5=-5A\Rightarrow A=1$
$\therefore \dfrac{5}{-x^2-x+6}=\dfrac{1}{x+3}-\dfrac{1}{x-2}$

6 $\dfrac{10x-1}{8x^2+2x-1}=\dfrac{10x-1}{(4x-1)(2x+1)}=\dfrac{A}{4x-1}+\dfrac{B}{2x+1}$

$$\Rightarrow 10x-1 = A(2x+1)+B(4x-1)$$

Set $x=-\frac{1}{2}$: $\quad -6=-3B \Rightarrow B=2$

Set $x=\frac{1}{4}$: $\quad \frac{3}{2}=\frac{3}{2}A \Rightarrow A=1$

$$\therefore \frac{10x-1}{8x^2+2x-1}=\frac{1}{4x-1}+\frac{2}{2x+1}$$

7 $\dfrac{11+3x}{6x^2+5x-6}=\dfrac{11+3x}{(3x-2)(2x+3)}=\dfrac{A}{3x-2}+\dfrac{B}{2x+3}$

$$\Rightarrow 11+3x = A(2x+3)+B(3x-2)$$

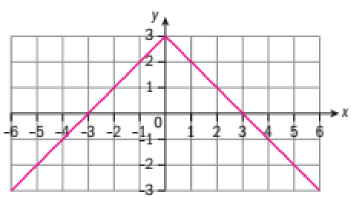
Set $x=\frac{2}{3}$: $\quad 13=\frac{13}{3}A \Rightarrow A=3$

Set $x=-\frac{3}{2}$: $\quad \frac{13}{2}=-\frac{13}{2}B \Rightarrow B=-1$

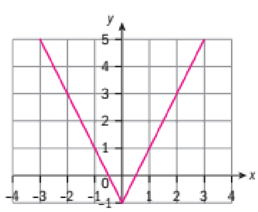
$$\therefore \frac{11+3x}{6x^2+5x-6}=\frac{3}{3x-2}-\frac{1}{2x+3}$$

Exercise 2G

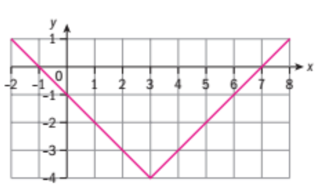
1 $D_f=\mathbb{R}$, $R_f=\{y\in\mathbb{R} \mid y\le 3\}$



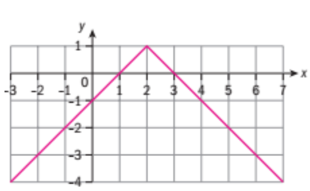
2 $D_f=\mathbb{R}$, $R_f=\{y\in\mathbb{R} \mid y\ge -1\}$



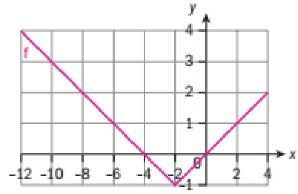
3 $D_f=\mathbb{R}$, $R_f=\{y\in\mathbb{R} \mid y\ge -4\}$



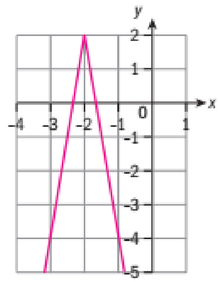
4 $D_f=\mathbb{R}$, $R_f=\{y\in\mathbb{R} \mid y\le 1\}$



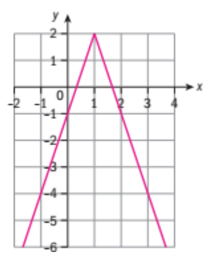
5 $D_f = \mathbb{R}$, $R_f = \{y \in \mathbb{R} \mid y \geq -1\}$



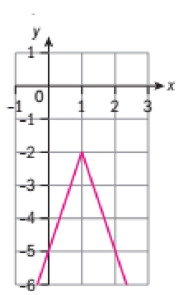
6 $D_f = \mathbb{R}$, $R_f = \{y \in \mathbb{R} \mid y \leq 2\}$



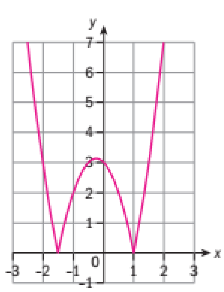
7 $D_f = \mathbb{R}$, $R_f = \{y \in \mathbb{R} \mid y \leq 2\}$



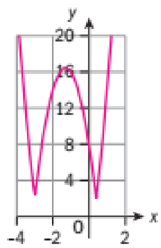
8 $D_f = \mathbb{R}$, $R_f = \{y \in \mathbb{R} \mid y \leq -2\}$



9 $D_f = \mathbb{R}$, $R_f = \{y \in \mathbb{R} \mid y \geq 0\}$



10 $D_f = \mathbb{R}$, $R_f = \{y \in \mathbb{R} \mid y \geq 2\}$



Exercise 2H

1 a $10-|3x+2|=7 \Rightarrow |3x+2|=3$

$\Rightarrow 3x+2=3$ or $3x+2=-3$

$3x+2=3 \Rightarrow x=\frac{1}{3}$

$3x+2=-3 \Rightarrow x=-\frac{5}{3}$

Substituting into the equation shows these are both valid

b $8|x+7|-3=5 \Rightarrow |x+7|=1$

$\Rightarrow x+7=1$ or $x+7=-1$

$x+7=1 \Rightarrow x=-6$

$x+7=-1 \Rightarrow x=-8$

Substituting into the equation show these are both valid

c $|x-2|=2x+1$

$\Rightarrow x-2=2x+1$ or $x-2=-(2x+1)$

$\Rightarrow x=-3$ or $x=\frac{1}{3}$

Substituting these into the equation $\Rightarrow x=\frac{1}{3}$ only

d $|4x+3|=3-x$

$\Rightarrow 4x+3=3-x$ or $4x+3=x-3$

$4x+3=3-x \Rightarrow x=0$

$4x+3=x-3 \Rightarrow x=-2$

Substituting these shows these are both valid

e $|4x+9|=|2x-1|$

$\Rightarrow 4x+9=2x-1$ or $4x+9=1-2x$

$4x+9=2x-1 \Rightarrow x=-5$

$4x+9=1-2x \Rightarrow x=-\frac{4}{3}$

Substituting these into the equation shows these are both valid

f $|5x+3|-|2x-1|=0 \Rightarrow |5x+3|=|2x-1|$

$\therefore 5x+3=2x-1$ or $5x+3=1-2x$

$5x+3=2x-1 \Rightarrow x=-\frac{4}{3}$

$5x+3=1-2x \Rightarrow x=-\frac{2}{7}$

Substituting these into the equation shows these are both valid

g $\left|\frac{2x-5}{3}\right|=\left|\frac{3x+4}{2}\right| \Rightarrow 2|2x-5|=3|3x+4|$

$\therefore 2(2x-5)=3(3x+4)$ or $2(2x-5)=-3(3x+4)$

$2(2x-5)=3(3x+4) \Rightarrow x=-\frac{22}{5}$

$2(2x-5)=-3(3x+4) \Rightarrow x=-\frac{2}{13}$

Substituting these into the equation shows these are both valid

Exercise 2I

1 a For $x>-\frac{3}{2}$,

$|2x+3|<6 \Rightarrow 2x+3<6 \Rightarrow x<\frac{3}{2}$

For $x<-\frac{3}{2}$, $|2x+3|<6 \Rightarrow -(2x+3)<6 \Rightarrow x>-\frac{9}{2}$

$\therefore -\frac{9}{2}<x<\frac{3}{2}$

b For $x \ge \frac{3}{2}$,

$|2x-3| \ge 5 \Rightarrow 2x-3 \ge 5 \Rightarrow x \ge 4$

For $x \le \frac{3}{2}$,

$|2x-3| \ge 5 \Rightarrow 3-2x \ge 5 \Rightarrow x \le -1$

$\therefore x \le -1$ or $x \ge 4$

c For $x<\frac{3}{2}$,

$|3-2x|<5 \Rightarrow 3-2x<5 \Rightarrow x>-1$

For $x>\frac{3}{2}$,

$|3-2x|<5 \Rightarrow 2x-3<5 \Rightarrow x<4$

$\therefore -1<x<4$

d $|1-3x| \ge 5$

For $x<\frac{1}{3}$, $1-3x \ge 5 \Rightarrow x \le -\frac{4}{3}$

For $x>\frac{1}{3}$, $3x-1 \ge 5 \Rightarrow x \ge 2$

$\therefore x \le -\frac{4}{3}$ or $x \ge 2$

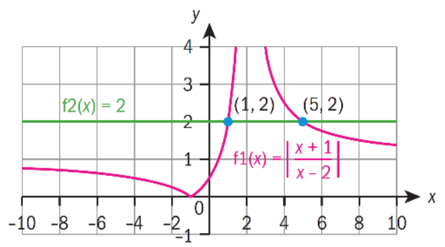
e $|2x+3| > |x+3|$

$2x+3 > x+3$ or $-(2x+3) > x+3$
$2x+3 > x+3 \Rightarrow x > 0$
$-(2x+3) > x+3 \Rightarrow x < -2$
Checking points in these regions shows they are both valid
$\therefore x < -2$ or $x > 0$

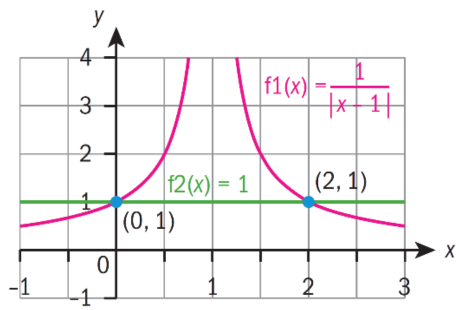
f $x+6 > |3x+2|$

$\Rightarrow x+6 > 3x+2$ or $x+6 > -(3x+2)$
$x+6 > 3x+2 \Rightarrow x < 2$
$x+6 > -(3x+2) \Rightarrow x > -2$
$\therefore -2 < x < 2$

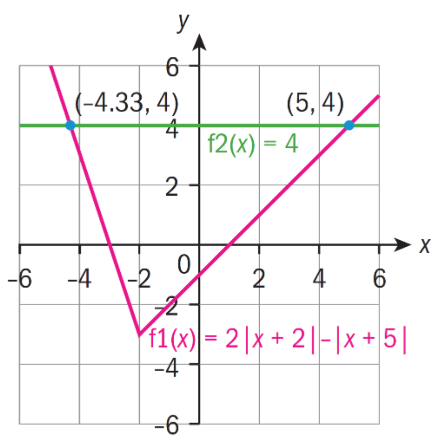
2 a From the graph, $1 < x < 5,\ x \neq 2$



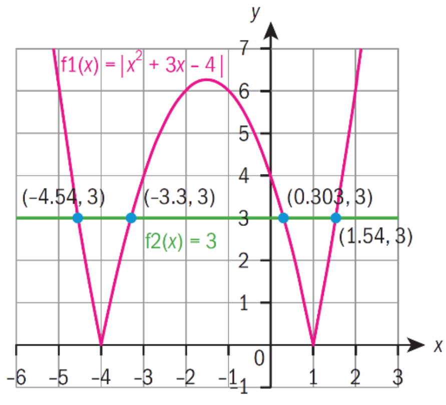
b From the graph, $0 < x < 2,\ x \neq 1$



c From the graph, $-\dfrac{13}{3} \le x \le 5,$



d From the graph, $-4.54 < x < -3.30$ and $0.303 < x < 1.54$



Exercise 2J

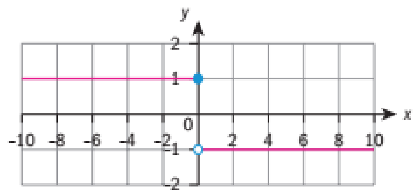
1 a $f(-9) = 1$

$f(0) = 1$

$f(\pi) = -1$

$f(99) = -1$

b



c $D_f = \mathbb{R}$

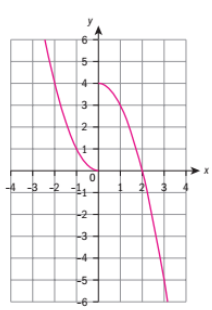
$R_f = \{-1, 1\}$

2 a $f(-4) = 16$

$f(0) = 0$

$f(1) = 3$

b



c $D_f = \mathbb{R}$

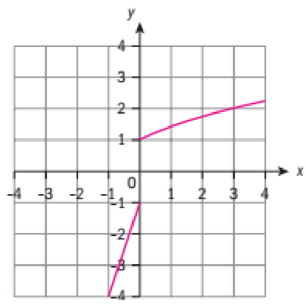
$R_f = \mathbb{R}$

3 **a** $f(-1) = -4$

$f(0) = -1$

$f(8) = 3$

b



c $D_f = \mathbb{R}$

$R_f = \{y \in \mathbb{R} \mid y > 1 \text{ or } y \le -1\}$

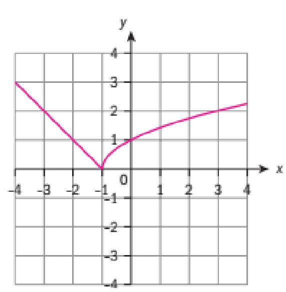
4 **a** $f(-1) = 0$

$f(0) = 1$

$f(-4) = 3$

$f(8) = 3$

b



c $D_f = \mathbb{R}$

$R_f = \{y \in \mathbb{R} \mid y \ge 0\}$

5 $f(x) = \begin{cases} 3x + 10, & x \le -2 \\ 2, & -2 < x < 2 \\ -3x + 10, & x \ge 2 \end{cases}$

6 **a** $f(x) = \begin{cases} 2x + 4, & x \ge -2 \\ -(2x + 4), & x < -2 \end{cases}$

b $f(x)=\begin{cases}3x-7, & x\ge 3\\ 11-3x, & x<3\end{cases}$

Exercise 2K

1 Neither

2 Onto and one-to-one □

3 One-to-one, not onto

4 One-to-one, not onto

5 Onto, not one-to-one

6 Onto, not one-to-one

7 Individual Response

Exercise 2L

1 **a** $f(-x)=2-(-x)^2=2-x^2=f(x)$ so even

Many-to-one

b $g(-x)=3(-x)+(-x)^3=-(3x+x^3)=-g(x)$ so odd

One-to-one

c $h(-x)=-\dfrac{1}{2(-x)}=\dfrac{1}{2x}=-\left(-\dfrac{1}{2x}\right)=-h(x)$ so odd

One-to-one

d $p(-x)=2(-x-3)^2$ so neither odd nor even

Many-to-one

e $f(x)$ isn't even defined for $x<0$ so neither

Many-to-one

f $f(-x)=(-x)-2(-x)^3+(-x)^5=-(x-2x^3+x^5)=-f(x)$ so odd

Many-to-one

2 Suppose that $f(x)$ is both even and odd

Then $f(-x)=f(x)=-f(x)$
$\Rightarrow 2f(x)=0$ for all x
$\Rightarrow f(x)=0$ for all x

Exercise 2M

1 a i $g(f(1)) = g(3) = \sqrt{3}$

ii $f(g(2)) = f(\sqrt{2}) = 3\sqrt{2}$

iii $f(g(x)) = f(\sqrt{x}) = 3\sqrt{x}$

iv $g(f(x)) = g(3x) = \sqrt{3x}$

b i $g(f(1)) = g(2) = 8$

ii $f(g(2)) = f(8) = -19$

iii $f(g(x)) = f(x^2 + 4) = 5 - 3(x^2 + 4) = -3x^2 - 7$

iv $g(f(x)) = g(5 - 3x) = 4 + (5 - 3x)^2 = 29 - 30x + 9x^2$

c i $g(f(1)) = g(2) = \sqrt{3}$

ii $f(g(2)) = f(\sqrt{3}) = \sqrt{3} + 1$

iii $f(g(x)) = f(\sqrt{2x - 1}) = \sqrt{2x - 1} + 1$

iv $g(f(x)) = g(x + 1) = \sqrt{2x + 1}$

2 a i $D_f = \mathbb{R}$

$R_f = \left\{y \in \mathbb{R} \mid y \geq -\frac{1}{4}\right\}$

$D_g = \mathbb{R}$

$R_g = \mathbb{R}$

ii $D_f = \mathbb{R}$

$R_f = \{y \in \mathbb{R} \mid y \geq 0\}$

$D_g = \{x \in \mathbb{R} \mid |x| \geq 2\}$

$R_g = \{y \in \mathbb{R} \mid y \geq 0\}$

b i $f \circ g(x) = (2 - 3x)^2 + (2 - 3x) = (2 - 3x)[(2 - 3x) + 1] = 3(x - 1)(3x - 2)$

$g \circ f(x) = 2 - 3(x^2 + x) = 2 - 3x - 3x^2$

$D_{f \circ g} = \mathbb{R}$

$R_{f \circ g} = \left\{y \in \mathbb{R} \mid y \geq -\frac{1}{4}\right\}$

$D_{g\circ f} = \mathbb{R}$

$R_{g\circ f} = \left\{y \in \mathbb{R} \mid y \le \frac{11}{4}\right\}$

ii $f \circ g(x) = \left|\sqrt{x^2 - 4} + 1\right|$

$g \circ f(x) = \sqrt{(x+1)^2 - 4} = \sqrt{(x+3)(x-1)}$

$D_{f\circ g} = \{x \in \mathbb{R} \mid |x| \ge 2\}$

$R_{f\circ g} = \{y \in \mathbb{R} \mid y \ge 1\}$

$D_{g\circ f} = \{x \in \mathbb{R} \mid x \le -3 \text{ or } x \ge 1\}$
$R_{g\circ f} = \{y \in \mathbb{R} \mid y \ge 0\}$

3 a i $f(h(x)) = 1 - 2\sqrt{2x+4}$

ii $h(g(x)) = \sqrt{2(x^2-1)+4} = \sqrt{2x^2+2}$

iii $h(h(x)) = \sqrt{2\sqrt{2x+4}+4}$

iv $f(g(h(x))) = f\left(\left(\sqrt{2x+4}\right)^2 - 1\right) = f(2x+3) = 1 - 2(2x+3) = -4x - 5$

b i $D_{f\circ h} = \{x \in \mathbb{R} \mid x \ge -2\}$, $R_{f\circ h} = \{y \in \mathbb{R} \mid y \le 1\}$

ii $D_{h\circ g} = \mathbb{R}$, $R_{h\circ g} = \left\{y \in \mathbb{R} \mid y \ge \sqrt{2}\right\}$

iii $D_{h\circ h} = \{x \in \mathbb{R} \mid x \ge -2\}$, $R_{h\circ h} = \{y \in \mathbb{R} \mid y \ge 2\}$

iv $D_{f\circ g\circ h} = \mathbb{R}$, $R_{f\circ g\circ h} = \mathbb{R}$

c $h(h(0)) = \sqrt{2\sqrt{2\times 0+4}+4} = \sqrt{2\sqrt{4}+4} = 2\sqrt{2}$

$g(g(x)) = x^2(x^2-2) \Rightarrow g(g(-1)) = (-1)^2\left((-1)^2 - 2\right) = -1$

4 $f(x) = 3x + a,\ g(x) = \frac{x-4}{3}$

$f(g(x)) = 3\cdot\frac{x-4}{3} + a = x - 4 + a$
$g(f(x)) = \frac{3x+a-4}{3} = x + \frac{a-4}{3}$
$\Rightarrow \frac{a-4}{3} = a - 4 \Rightarrow a - 4 = 3a - 12$
$\Rightarrow 2a = 8 \Rightarrow a = 4$

5 Individual Response

6 a $(b \circ t)(h) = 20(4h+2)^2 - 80(4h+2) + 500$

$= 20\left(16h^2 + 16h + 4\right) - 320h - 160 + 500$
$= 320h^2 + 420$

This gives the number of bacteria b in food h hours out of the refrigerator.

b $10000 = 320h^2 + 420$

h = 5.47 hours

7 $r(t) = r(v(t)) = \left(\dfrac{40 + 3t + t^2}{500} - 0.1\right)^2 + 0.2$; 2 hours

Exercise 2N

1 a {(2.4),(2,0),(2,−2,),(2,2)}. Inverse relation is not a function since 2 has more than one image.

b $\{(3,1),\ (2,-6),\ (-4,-3),\ (0,0),\ (-5,-5),\ (-3,-2)\}$

c $\{(-1,-1),\ (3,-3),\ (-5,-2),\ (-4,-4),\ (1,1),\ (3,-5),\ (-2,0)\}$

2 a $x = 5y - 1 \Rightarrow y = f^{-1}(x) = \dfrac{x+1}{5}$

b $x = \dfrac{y-2}{3} \Rightarrow y = f^{-1}(x) = 3x + 2$

c $x = y^2 - 3 \Rightarrow y = f^{-1}(x) = \sqrt{x+3}$

(must restrict to either positive or negative square root for this to be a function)

d $x = \dfrac{2}{y-3}\ (y \neq 3) \Rightarrow xy - 3x = 2 \Rightarrow y = f^{-1}(x) = \dfrac{2+3x}{x}\ (x \neq 0)$

e $x = y^3 + 1 \Rightarrow y = f^{-1}(x) = (x-1)^{\frac{1}{3}}$

f $x = \dfrac{y+1}{y-1}\ (y \neq 1)$

$xy - x = y + 1 \Rightarrow y(x-1) = x + 1$
$\Rightarrow y = f^{-1}(x) = \dfrac{x+1}{x-1}\ (x \neq 1)$

3 a To make f a function, restrict the domain:

$D_f = \{x \in \mathbb{R} \mid x \geq 2\}$
$R_f = \{y \in \mathbb{R} \mid y \geq 0\}$

$x = (y-2)^2 \Rightarrow \pm\sqrt{x} = y - 2 \Rightarrow y = 2 \pm \sqrt{x}$

Take positive square root to make this a function, and restrict domain to $x \geq 0$

$\Rightarrow y = f^{-1}(x) = 2 + \sqrt{x}$

$D_{f^{-1}} = \{x \in \mathbb{R} \mid x \geq 0\}$

$R_{f^{-1}} = \{y \in \mathbb{R} \mid y \geq 2\}$

b The domain of the original function does not need restricting:

$D_f = \{x \in \mathbb{R} \mid x \neq -1\}$

$R_f = \{y \in \mathbb{R} \mid y \neq 2\}$

$$x = \frac{2y+1}{y+1}, \quad y \neq -1$$

$xy + x = 2y + 1 \Rightarrow y(2 - x) = x - 1$

$\Rightarrow y = f^{-1}(x) = \dfrac{x-1}{2-x}, \quad x \neq 2$

$D_{f^{-1}} = \{x \in \mathbb{R} \mid x \neq 2\}$

$R_{f^{-1}} = \{y \in \mathbb{R} \mid y \neq -1\}$

c Restrict the domain so that it is a function:

$D_f = \{x \in \mathbb{R} \mid x \geq 0\}$

$R_f = \{y \in \mathbb{R} \mid y \geq 1\}$

$$x = 4y^2 + 1 \Rightarrow y^2 = \frac{x-1}{4} \Rightarrow y = \pm\frac{\sqrt{x-1}}{2}$$

Take e.g. positive square root to make this a function and restrict domain such that $x \geq 1$

$y = f^{-1}(x) = \dfrac{\sqrt{x-1}}{2}$

therefore

$D_{f^{-1}} = \{x \in \mathbb{R} \mid x \geq 1\}$

$R_{f^{-1}} = \{y \in \mathbb{R} \mid y \geq 0\}$

4 This can be done by direct substitution, but note that in general,

$(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1} = g \circ (\mathrm{id}) \circ g^{-1} = g \circ (\mathrm{id} \circ g^{-1}) = g \circ g^{-1} = \mathrm{id}$

where id is the identity function $\mathrm{id}(x) = x$

$\Rightarrow (g \circ f) \circ (f^{-1} \circ g^{-1})(x) = x$

$\Rightarrow (f^{-1} \circ g^{-1})(x) = (g \circ f)^{-1}(x)$

Since this is true in general, it is certainly true for the specified functions

5 Important: it must be shown that both $f(g(x)) = x$ and $g(f(x)) = x$

a $f(g(x)) = -4\left(1 - \dfrac{x}{4}\right) + 4 = -4 + x + 4 = x$

and $g(f(x)) = 1 - \dfrac{1}{4}(-4x + 4) = 1 + x - 1 = x$

b $f\left(g\left(x\right)\right)=\dfrac{\dfrac{2}{1-x}+3-5}{\dfrac{2}{1-x}+3-3}=\dfrac{2-2\left(1-x\right)}{2}=x$

and $g\left(f\left(x\right)\right)=-\dfrac{2}{\dfrac{x-5}{x-3}-1}+3=-\dfrac{2\left(x-3\right)}{x-5-\left(x-3\right)}+3=\left(x-3\right)+3=x$

c $f\left(g\left(x\right)\right)=\dfrac{\left(2\dfrac{\sqrt[3]{2x}-3}{2}+3\right)^3}{2}=\dfrac{\left(\sqrt[3]{2x}\right)^3}{2}=\dfrac{2x}{2}=x$

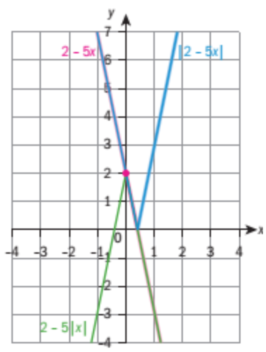
and $g\left(f\left(x\right)\right)=\dfrac{\sqrt[3]{2\dfrac{\left(2x+3\right)^3}{2}}-3}{2}=\dfrac{\left(2x+3\right)-3}{2}=x$

Exercise 2O

For all of **a**, to transform $y=f(x)$ to $y=|f(x)|$, the graph is unchanged for $y\geq 0$, and reflected in the x=axis for $y<0$.

For all of **ii**, to transform $y=f(x)$ to $y=f(|x|)$, the graph is unchanged for $x\geq 0$. Where $x<0$, the part of the graph for $x\geq 0$ is reflected in the y-axis.

1



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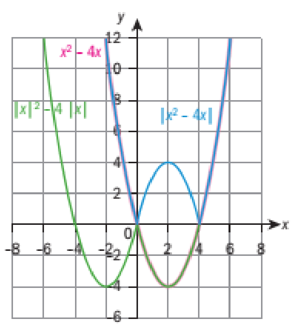

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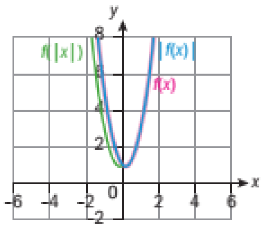

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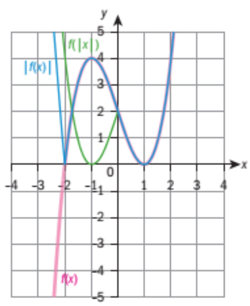

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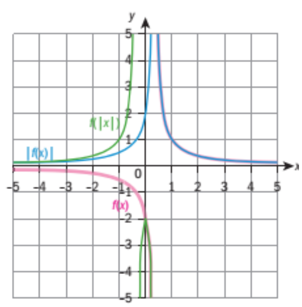

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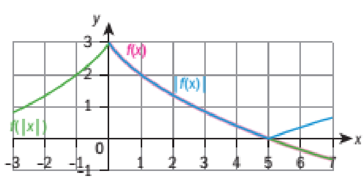
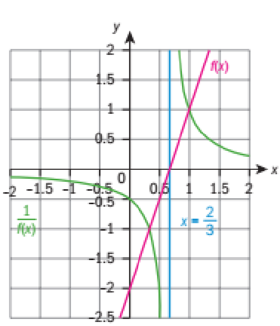
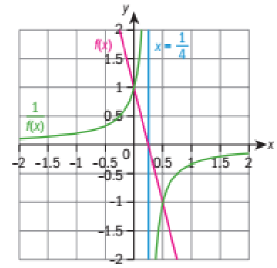


Exercise 2P

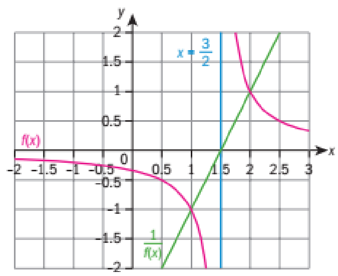
1 a



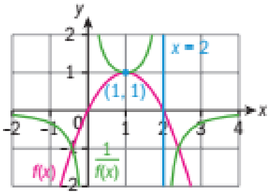
b


f(x)
$x = \frac{1}{4}$
$\frac{1}{f(x)}$
x
y

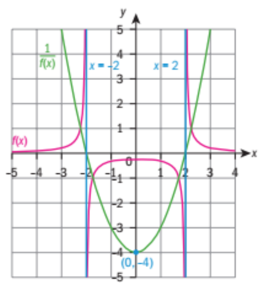
c


$x = \frac{3}{2}$
f(x)
$\frac{1}{f(x)}$
x
y

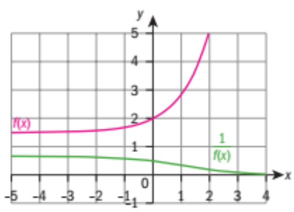
d


x = 2
(1, 1)
f(x)
$\frac{1}{f(x)}$
x
y

e

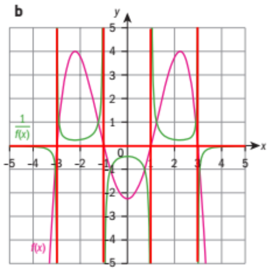

$\frac{1}{f(x)}$
x = -2
x = 2
f(x)
(0, -4)
x
y

2 a


f(x)
$\frac{1}{f(x)}$
x
y

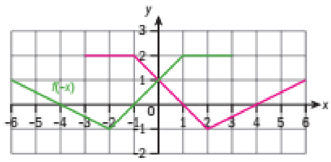
b

b

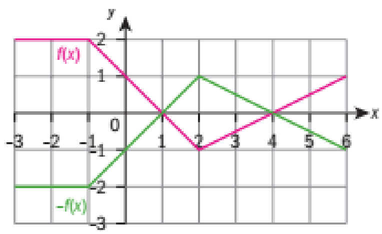


Exercise 2Q

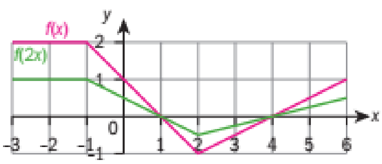
1 a



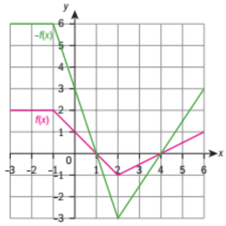
b



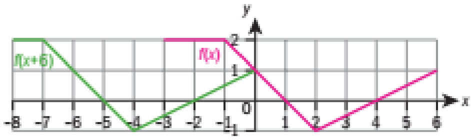
c



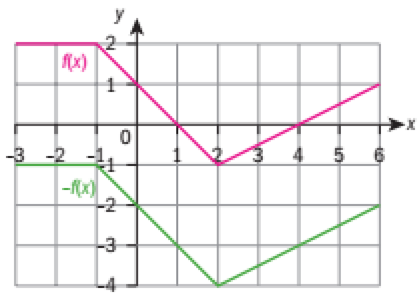
d



e



f



2 **a** $r(x) = 2f(x)$

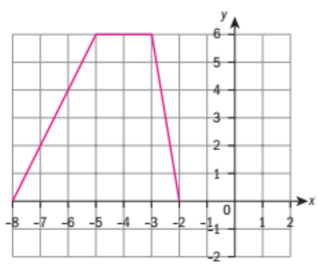
$s(x) = -f(x-3)$

b $r(x) = f(-x)$

$s(x) = f\left(\dfrac{x}{2}\right) - 4$

3 **a** $R_f = \{y \in \mathbb{R} \mid 0 \le y \le 6\}$

b



c $D_g = \{x \in \mathbb{R} \mid -8 \le x \le -2\}$

d $h(x) = g(x) - 4$

e $h(x) = f(-x) - 4$

4 **a** $g(x) = -f(x)$

b $g(x) = f(-x)$

c $g(x) = f(x+3) - 1$

d $g(x) = -f(x) + 1$

e $g(x) = \dfrac{1}{f(x)}$

f $g(x) = -f(2x)$

Exercise 2R

1 In order of the transformations given, the function $y = \frac{1}{x}$ is transformed to

$y = \frac{2}{x}$

then $y = \frac{2}{3x}$

then $y = \frac{2}{3(x+2)} + 3$

$D_f = \{x \in \mathbb{R} \mid x \neq -2\}$, $R_f = \{y \in \mathbb{R} \mid y \neq 3\}$

2 **a** Horizontal dilation factor of $\frac{1}{3}$, followed by a vertical dilation of factor 2, then a horizontal translation of 4 units in the positive x-direction, and a vertical translation of 1 unit in the positive y-direction.

b $y = 2f\left(3(x-4)\right) + 1$

3 **a** $y = \frac{x-3}{x+5} = \frac{x+5-8}{x+5} = 1 - \frac{8}{x+5}$

e.g. translation by $\begin{pmatrix} -5 \\ 0 \end{pmatrix}$, vertical stretch by factor -8, translation by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

b $y = \frac{4x+5}{2x+1} = \frac{2(2x+1)+3}{2x+1} = 2 + \frac{3}{2x+1}$

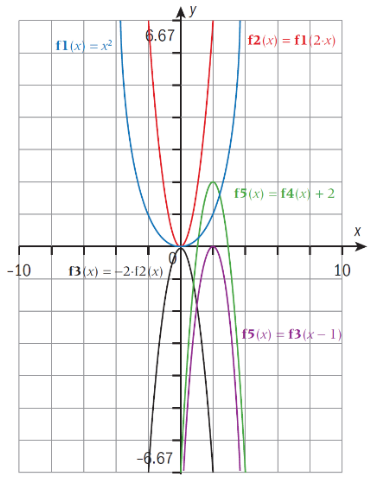
e.g. translation by $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$, stretch horizontally by scale factor $\frac{1}{2}$, stretch vertically by scale factor 3, translation by $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

c $y = \frac{2x+4}{x+1} = \frac{2(x+1)+2}{x+1} = 2 + \frac{2}{x+1}$

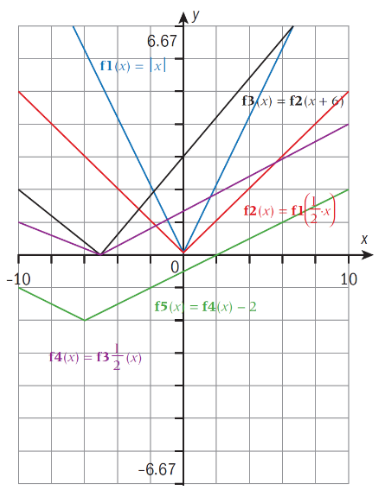
e.g. translation by $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$, stretch vertically by scale factor 2, translation by $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

Exercise 2S

1



2



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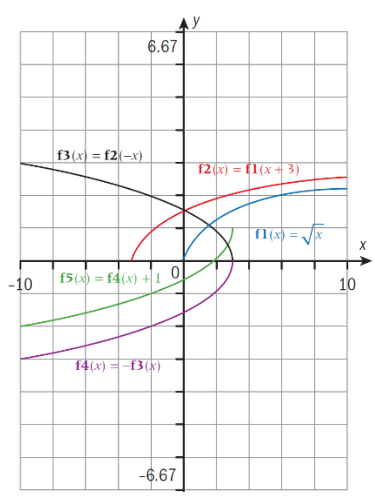
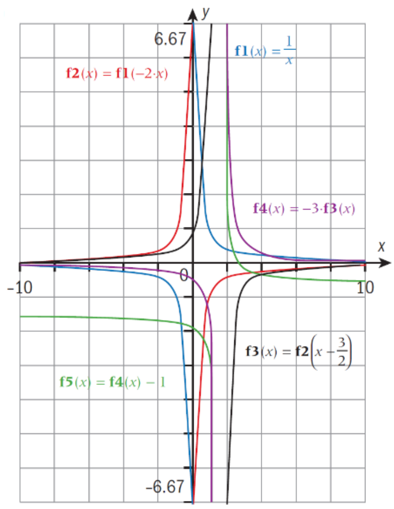
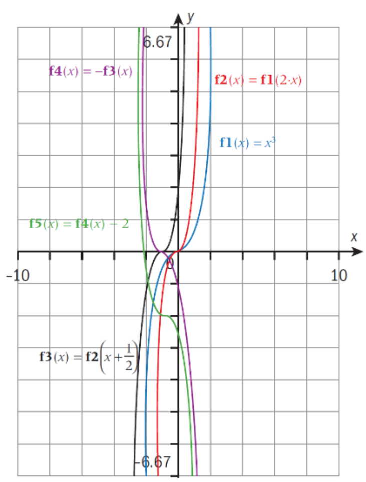


4



5



Chapter review

1 **a** The graph does not represent a function since it does not pass the vertical line test.

b This graph represents a function.

$D_f = \{-4 \le x \le 5\}$ $\quad R_f = \{-1, 2\}$

c This graph represents a function.

$D_f = \mathbb{R}$ $\quad R_f = \{y \in \mathbb{R} \mid -1 \le y \le 1\}$

d This mapping does not represent a function since 5 is mapped on to more than one element.

e This graph does not represent a function since it does not pass the vertical line test.

f This relation represents a function.

$D_f = \{-1, 0, 3, \pi\}$

$R_f = \{\pi\}$

2 **a** $f\left(g\left(h(2)\right)\right)=f\left(g(3)\right)=f(1)=2$

b $h^{-1}\left(g^{-1}\left(f^{-1}(2)\right)\right)=h^{-1}\left(g^{-1}(1)\right)=h^{-1}(3)=2$

3 **a** $x=\dfrac{y-2}{5}\Rightarrow y=f^{-1}(x)=5x+2$

b $x=\sqrt{1-y}\Rightarrow x^2=1-y\Rightarrow y=g^{-1}(x)=1-x^2$

c $x=\dfrac{3y}{2-y}\Rightarrow 2x-xy=3y\Rightarrow y=\dfrac{2x}{3+x}\quad (x\neq -3)$

4 Translate the graph of y=f(x) 3 units in the negative x-direction, reflect in the y-axis, vertical stretch by a factor of 2, vertical translation of 4 units in the positive direction.

5 $x-2=2x+1\Rightarrow x=-3$

$2-x=2x+1\Rightarrow x=\dfrac{1}{3}$

Graphical representation $\Rightarrow$ the desired region is the section between these intersections

i.e. $-3\leq x\leq \dfrac{1}{3}$

6 $x=\dfrac{1}{1+y^2}\Rightarrow x+xy^2=1\Rightarrow y^2=\dfrac{1-x}{x}$

Take $y=\sqrt{\dfrac{1-x}{x}}$

$$\frac{1}{1+\left(\sqrt{\dfrac{1-x}{x}}\right)^2}=\frac{1}{1+\dfrac{1-x}{x}}=\frac{x}{x+1-x}=x$$

and $\sqrt{\dfrac{1-\dfrac{1}{1+x^2}}{\dfrac{1}{1+x^2}}}=\sqrt{\dfrac{1+x^2-1}{1}}=\sqrt{x^2}=|x|=x$ in the domain $[0,1]$

7 **a** $y=f(x)=\dfrac{x^2+1}{|x|}$

$f(-x)=\dfrac{(-x)^2+1}{|-x|}=\dfrac{x^2+1}{|x|}=f(x)$ so even

b $y=f(x)=\dfrac{x}{x^2+1}$

$f(-x)=\dfrac{(-x)}{(-x)^2+1}=-\dfrac{x}{x^2+1}=-f(x)$ so odd

c $f(x)=\dfrac{\sqrt{x}}{x};f(-x)=-\dfrac{\sqrt{-x}}{x};-f(x)=-\dfrac{\sqrt{x}}{x}$, hence neither.

8 $\dfrac{2}{x^2+5x+6}=\dfrac{2}{(x+3)(x+2)}=\dfrac{A}{x+2}+\dfrac{B}{x+3}$

$\Rightarrow 2 = A(x+3) + B(x+2)$

Set $x = -2$: $2 = A \Rightarrow A = 2$

Set $x = -3$: $2 = -B \Rightarrow B = -2$

$\therefore \dfrac{2}{x^2+5x+6} = \dfrac{2}{x+2} - \dfrac{2}{x+3}$

Exam-style questions

9 a No real roots $\Rightarrow \Delta < 0$ (1 mark)

$\Delta = 36 - 4(2k)(k) = 36 - 8k^2$ (1 mark)

$36 - 8k^2 < 0$

$k^2 > \dfrac{36}{8} = \dfrac{9}{2}$

$|k| > \dfrac{3}{\sqrt{2}}$

$k < -\dfrac{3}{\sqrt{2}}$ or $k > \dfrac{3}{\sqrt{2}}$ (2 marks)

b Equation of line of symmetry is $x = -\dfrac{b}{2a} = -\dfrac{6}{4k} = -\dfrac{3}{2k}$ (2 marks)

Therefore $\dfrac{3}{2k} = 1$

$\Rightarrow k = \dfrac{3}{2}$ (1 mark)

10 a The graph of f is shifted two units in the positive x-direction and one unit in the negative y-direction.

b $y = \left(2(x-2)^2 + 4(x-2) + 7\right) - 1$ (3 marks)

$= \left(2(x^2 - 4x + 4) + 4x - 8 + 7\right) - 1$

$= 2x^2 - 4x + 6$ (1 mark)

11 a $x = \dfrac{3y-4}{y+2} \Rightarrow yx + 2x = 3y - 4$ (2 marks)

$yx - 3y = -4 - 2x$

$y(x-3) = -(4+2x)$

$y = \dfrac{2x+4}{3-x}$

$f^{-1}(x) = \dfrac{2x+4}{3-x}$ (1 mark)

b $x \neq 3$ (1 mark)

12 a $y = \dfrac{k}{x-1} + 1$

$x = \dfrac{k}{y-1} + 1$

$x(y-1) = k + y - 1$ (1 mark)

$xy - x = k + y - 1$ (1 mark)

$xy - y = k + x - 1$

$y(x-1) = k + x - 1$

$y = \dfrac{k + x - 1}{x-1}$

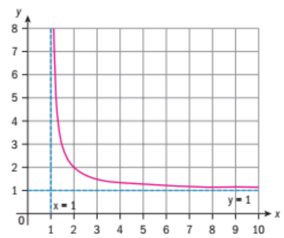
$y = \dfrac{k}{x-1} + 1$ (1 mark)

$f^{-1}(x) = \dfrac{k}{x-1} + 1$

So f is self-inverse

b Range is $f(x) > 1$, $f(x) \in$ (2 marks)

c



(shape of graph: 1 mark)

(both asymptotes: 1mark)

13 a $x^2 - 6x + 13 = (x-3)^2 + 4$ (2 marks)

Therefore $k = 3$ (1 mark)

b $y = (x-3)^2 + 4$

$x = (y-3)^2 + 4$

$(y-3)^2 = x - 4$ (2 mark)

$y - 3 = \sqrt{x-4}$

$y = 3 + \sqrt{x-4}$

$f^{-1}(x) = 3 + \sqrt{x-4}$ (1 mark)

c The domain of $f^{-1}(x)$ is $x \geq 4$, $(x \in \mathbb{R})$ (1 mark)

The range of $f^{-1}(x)$ is $f(x) \geq 3$, $(f(x) \in \mathbb{R})$ (1 mark)

14 a $f(x) = \dfrac{17-10x}{2x-1} = \dfrac{12+5-10x}{2x-1}$ (2 marks)

$= \dfrac{12+5(1-2x)}{2x-1}$ (1 mark)

$= \dfrac{12-5(2x-1)}{2x-1}$

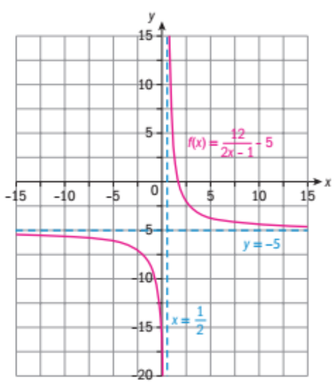
$= \dfrac{12}{2x-1} - \dfrac{5(2x-1)}{(2x-1)}$

$= \dfrac{12}{2x-1} - 5$ (1 mark)

b i $x = \dfrac{1}{2}$ (1 mark)

ii $y = -5$ (1 mark)

c



(1 mark for each branch correctly drawn, 1 mark for both asymptotes; 3 marks total)

15 a 6 (1 mark)

b $P = \dfrac{18(1+0.82\times 12)}{3+(0.034\times 12)} \approx 57$ (2 marks)

c Solving $100 = \dfrac{18(1+0.82t)}{3+0.034t}$ (1 mark)

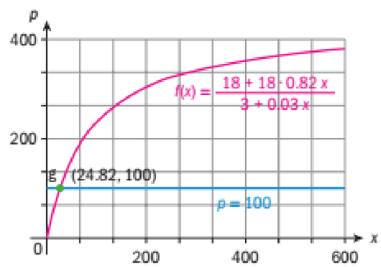
$300 + 3.4t = 18(1+0.82t)$

$300 + 3.4t = 18 + 14.76t$

$282 = 11.36t$

$t = \dfrac{282}{11.36} = 24.8$ months (1 mark)

OR



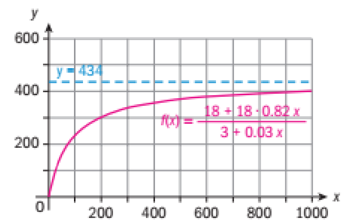
(1 mark)

t = 24.8 months (1 mark)

d A horizontal asymptote exists at $P = \dfrac{18 \times 0.82}{0.034} = 434.12$ (2 marks)

Therefore for $t \geq 0$, $P \leq 434$ (1 mark)

OR



(1 mark)

A horizontal asymptote exists at P = 434 (1 mark)

Therefore for $t \geq 0$, $P < 434$, $P < 434$ (1 mark)

16 $(x-3)^2 = x^2 - 6x + 9$ (1 mark)

$2(x-3)^2 = 2x^2 - 12x + 18$ (2 marks)

$2(x-3)^2 + 12x = 2x^2 + 18$

$2(x-3)^2 + 12(x-3) = 2x^2 + 18 - 36$

$2(x-3)^2 + 12(x-3) = 2x^2 - 18$

Therefore $g(x) = 2x^2 + 12x$ (1 mark)

3 Expanding the number system: complex numbers

Skills check

1 **a** $x=\pm 13$ **b** $x=\pm 2\sqrt{7}$ **c** $x=\pm 3\sqrt{11}$

2 **a** x = 1, −2 **b** $x=-\frac{1}{3},1$

3 **a** $x<-3$ **b** $-4\le x\le 2$ **c** $-\frac{1}{4}\le x\le 2$

4 **a** $x = -5$, $y = -9$ **b** $y=\frac{1}{3}-\frac{2x}{3}$

c $x=\frac{13}{8},y=-\frac{3}{4}$ **d** No solutions

Exercise 3A

1 $x^2+8x+15=(x+5)(x+3)=0$

$\therefore x=-5$ or $x=-3$

2 $x^2+5x-14=(x+7)(x-2)=0$

$\therefore x=-7$ or $x=2$

3 $3x^2-7x+2=(3x-1)(x-2)=0$

$\therefore x=\frac{1}{3}$ or $x=2$

4 $4x^2-20x+25=(2x-5)^2=0$

$\therefore x=\frac{5}{2}$

5 $5x^2-4x-12=(5x+6)(x-2)=0$

$\therefore x=-\frac{6}{5}$ or $x=2$

Exercise 3B

1 **a** $x^2+6x-7=(x+3)^2-16=0$

$\therefore (x+3)^2=16\Rightarrow x=-3\pm 4$

$\therefore x=-7$ or $x=1$

b $x^2-7x-30=\left(x-\frac{7}{2}\right)^2-\frac{49}{4}-30=0$

$$\therefore \left(x-\frac{7}{2}\right)^2=\frac{169}{4} \Rightarrow x=\frac{7}{2}\pm\frac{13}{2}$$

$\therefore x=-3$ or $x=10$

c $x^2-x-1=\left(x-\frac{1}{2}\right)^2-\frac{1}{4}-1=0$

$$\therefore \left(x-\frac{1}{2}\right)^2=\frac{5}{4} \Rightarrow x=\frac{1\pm\sqrt{5}}{2}$$

d $x^2-\frac{7}{3}x+\frac{2}{3}=\left(x-\frac{7}{6}\right)^2-\frac{49}{36}+\frac{2}{3}=0$

$$\Rightarrow \left(x-\frac{7}{6}\right)^2=\frac{25}{36}$$

$$\Rightarrow x=\frac{7\pm5}{6} \Rightarrow x=\frac{1}{3} \text{ or } x=2$$

e $4x^2+12x+5=0$

$$x^2+3x+\frac{5}{4}=0$$

$$\therefore \left(x+\frac{3}{2}\right)^2-\frac{9}{4}+\frac{5}{4}=0$$

$$\Rightarrow \left(x+\frac{3}{2}\right)^2=1$$

$$\Rightarrow x=-\frac{3}{2}\pm1$$

$$\therefore x=-\frac{5}{2} \text{ or } x=-\frac{1}{2}$$

f $x^2-2x+\frac{2}{5}=(x-1)^2-\frac{3}{5}=0$

$$(x-1)^2=\frac{3}{5}$$

$$\therefore x=1\pm\sqrt{\frac{3}{5}}=\frac{\sqrt{5}\pm\sqrt{3}}{\sqrt{5}}=1\pm\frac{\sqrt{15}}{5}$$

2 a $x^2+2x-1=(x+1)^2-2=0$

$\Rightarrow x=1\pm\sqrt{2}$

$\therefore x=2.41$ or $x=-0.414$ to 3s.f.

b $x^2-3x+1=\left(x-\frac{3}{2}\right)^2+1-\frac{9}{4}=\left(x-\frac{3}{2}\right)^2-\frac{5}{4}=0$

$$\Rightarrow x=\frac{3\pm\sqrt{5}}{2}$$

$\therefore x=0.382$ or $x=2.62$ to 3s.f.

c $x^2-\frac{1}{2}x-\frac{3}{2}=0$

$$\left(x-\frac{1}{4}\right)^2-\frac{3}{2}-\frac{1}{16}=\left(x-\frac{1}{4}\right)^2-\frac{25}{16}=0$$

$$\Rightarrow x=\frac{1\pm 5}{4}$$

$$\therefore x=-1.00 \quad \text{or} \quad x=1.50 \text{ to 3s.f.}$$

d $x^2+3x+\frac{5}{3}=0$

$$\left(x+\frac{3}{2}\right)^2-\frac{9}{4}+\frac{5}{3}=\left(x+\frac{3}{2}\right)^2-\frac{7}{12}=0$$

$$\Rightarrow x=-\frac{3}{2}\pm\sqrt{\frac{7}{12}}$$

$$\therefore x=-2.26 \quad \text{or} \quad -0.736 \quad \text{to 3s.f.}$$

Exercise 3C

1 a $x^2+9x+18=0$

$$\therefore x=\frac{-9\pm\sqrt{9^2-4(1)(18)}}{2(1)}=\frac{-9\pm\sqrt{9}}{2}=\frac{-9\pm 3}{2}$$

$$\Rightarrow x=-6 \quad \text{or} \quad x=-3$$

b $x^2-x-30=0$

$$\therefore x=\frac{1\pm\sqrt{(-1)^2-4(1)(-30)}}{2(1)}=\frac{1\pm 11}{2}$$

$$\Rightarrow x=-5 \quad \text{or} \quad x=6$$

c $x^2-x+1=0$

$$\therefore x=\frac{1\pm\sqrt{(-1)^2-4(1)(1)}}{2(1)}=\frac{1\pm\sqrt{-3}}{2}\notin\mathbb{R}$$

d $2x^2-3x-2=0$

$$x=\frac{3\pm\sqrt{(-3)^2-4(2)(-2)}}{2(2)}=\frac{3\pm\sqrt{25}}{4}=\frac{3\pm 5}{4}$$

$$\therefore x=2 \quad \text{or} \quad x=-\frac{1}{2}$$

e $\sqrt{2}x^2-11x+\sqrt{3}=0$

$$\therefore x=\frac{11\pm\sqrt{(-11)^2-4\left(\sqrt{2}\right)\left(\sqrt{3}\right)}}{2\sqrt{2}}=\frac{11\pm\sqrt{121-4\sqrt{6}}}{2\sqrt{2}}$$

2 a $x^2+(3-a)x-3a=0$

$$\therefore x = \frac{-(3-a) \pm \sqrt{(3-a)^2 - 4(-3a)}}{2}$$
$$= \frac{a-3 \pm \sqrt{a^2+6a+9}}{2}$$
$$= \frac{a-3 \pm \sqrt{(a+3)^2}}{2}$$
$$= \frac{a-3 \pm |a+3|}{2}$$

so $x = a$ or $x = -3$

b $2x^2 + (2b-1)x - b = 0$

$$\therefore x = \frac{1-2b \pm \sqrt{(2b-1)^2 - 4(2)(-b)}}{2(2)}$$
$$= \frac{1-2b \pm \sqrt{4b^2+4b+1}}{4}$$
$$= \frac{1-2b \pm \sqrt{(2b+1)^2}}{4}$$
$$= \frac{1-2b \pm |2b+1|}{4}$$

so $x = \frac{1}{2}$ or $x = -b$

c $x^2 + kx - 2k^2 = 0$

$$x = \frac{-k \pm \sqrt{k^2 - 4(1)(-2k^2)}}{2}$$
$$= \frac{-k \pm \sqrt{9k^2}}{2} = \frac{-k \pm 3|k|}{2}$$

so $x = k$ or $x = -2k$

d $p^2x^2 + 2px - 3 = 0$

$$\therefore x = \frac{-2p \pm \sqrt{(2p)^2 - 4p^2(-3)}}{2p^2}$$
$$= \frac{-2p \pm \sqrt{16p^2}}{2p^2} = \frac{-2p \pm 4|p|}{2p^2}$$

so $x = -\frac{3}{p}$ or $x = \frac{1}{p}$

Exercise 3D

1 a $\Delta = 3^2 - 4(1)(-7) = 37 > 0$ so two distinct real roots

b $\Delta = 1^2 - 4(1)(2) = -7 < 0$ so no real roots

c $\Delta = (2)^2 - 4(1)(1) = 0$ so one repeated real root

d $\Delta = \left(\sqrt{3}\right)^2 - 4(5)(2) = -37 < 0$ so no real roots

e $2x^2 - \pi x + 1 = 0$

$\Delta = (-\pi)^2 - 4(2)(1) = \pi^2 - 8 > 3^2 - 8 = 1 > 0$ so two distinct real roots

f $2.25x^2 - 21x + 49 = 0$

$\Delta = (-21)^2 - 4(2.25)(49) = 0$ so one repeated real root

2 **a** $mx^2 + 2x - 5 = 0$

$\Delta = 2^2 - 4m(-5) = 4 + 20m$

Two distinct real roots: $4 + 20m > 0 \Rightarrow m > -\frac{1}{5}$

One repeated real root: $m = -\frac{1}{5}$

No real roots: $m < -\frac{1}{5}$

b $4x^2 - 3x + t - 4 = 0$

$\Delta = (-3)^2 - 4(4)(t-4) = 9 - 16(t-4) = 73 - 16t$

Two distinct real roots: $t < \frac{73}{16}$

One repeated real root: $t = \frac{73}{16}$

No real roots: $t > \frac{73}{16}$

c $(2s+1)x^2 = s(3x-1)$

$(2s+1)x^2 - 3sx + s = 0$

$\Delta = (-3s)^2 - 4s(2s+1) = 9s^2 - 8s^2 - 4s = s(s-4)$

Two distinct real roots: $s < 0$ or $s > 4$

One repeated real root: $s = 0$ or $s = 4$

No real roots: $0 < s < 4$

Exercise 3E

1 $(x+4)(x+2) < 0$

$\therefore x \in]-4, -2[$

2 $(x-4)^2 > 0$

$\therefore x \in \mathbb{R} \setminus \{4\}$

3 $(x-15)(x+2) \geq 0$

$\therefore x \in]-\infty, -2] \cup [15, \infty[$

4 $(4x-3)(x-2)\le 0$

$\therefore x \in \left[\frac{3}{4}, 2\right]$

5 $5x^2-6x-8>0$

$(5x+4)(x-2)>0$

$\therefore x \in \left]-\infty, -\frac{4}{5}\right[\cup \left]2, \infty\right[$

6 $9x^2-12x+4\le 0$

$(3x-2)^2 \le 0$ so $x \in \left\{\frac{2}{3}\right\}$

Exercise 3F

1 **a** $\text{Re}(z)=0,\ \text{Im}(z)=-4$

b $\text{Re}(z)=5,\ \text{Im}(z)=0$

c $\text{Re}(z)=-24,\ \text{Im}(z)=7$

d $\text{Re}(z)=\frac{5}{13},\ \text{Im}(z)=-\frac{12}{13}$

e $\text{Re}(z)=-\frac{1}{\sqrt{5}},\ \text{Im}(z)=\frac{2}{\sqrt{5}}$

2 **a** $|z|=\sqrt{0^2+(-4)^2}=\sqrt{16}=4$

b $|z|=\sqrt{5^2+0^2}=\sqrt{25}=5$

c $|z|=\sqrt{(-24)^2+7^2}=\sqrt{576+49}=\sqrt{625}=25$

d $|z|=\sqrt{\left(\frac{5}{13}\right)^2+\left(-\frac{12}{13}\right)^2}=\sqrt{\frac{169}{169}}=1$

e $|z|=\sqrt{\left(-\frac{1}{\sqrt{5}}\right)^2+\left(\frac{2}{\sqrt{5}}\right)^2}=1$

3 **a** $z_1-z_2+z_3=(3-5-1)+i(-4-1+2)=-3-3i$

b $2z_1+3z_2-4z_3=(6+15+4)+i(-8+3-8)=25-13i$

c $-\frac{1}{2}z_1+\frac{2}{3}z_2-\frac{1}{4}z_3$

$$-\frac{1}{2}(3-4i)+\frac{2}{3}(5+i)-\frac{1}{4}(-1+2i)$$
$$=\left(-\frac{3}{2}+\frac{10}{3}+\frac{1}{4}\right)+i\left(2+\frac{2}{3}-\frac{1}{2}\right)$$
$$=\frac{25}{12}+\frac{13i}{6}$$

d $\frac{4z_3-5z_1+2z_2}{3}$

$$=\frac{4(-1+2i)-5(3-4i)+2(5+i)}{3}$$
$$=\frac{-4+8i-15+20i+10+2i}{3}$$
$$=\frac{-9+30i}{3}=-3+10i$$

Exercise 3G

1 **a** $z_1z_2+2z_3^*=(1+i)(3-2i)+2(-2-3i)$

$$=5+i-4-6i=1-5i$$

b $\frac{z_1}{z_3}-\frac{z_2}{5}=\frac{z_1z_3^*}{|z_3|^2}-\frac{z_2}{5}$

$$=\frac{(1+i)(-2-3i)}{(-2)^2+3^2}-\frac{3-2i}{5}$$
$$=\frac{1-5i}{13}-\frac{3-2i}{5}$$
$$=\frac{5-25i-39+26i}{65}$$
$$=-\frac{34}{65}+\frac{1}{65}i$$

c $z_1^2-3z_2z_3=(1+i)^2-3(3-2i)(-2+3i)$

$$=1+2i-1-3(-6+13i+6)$$
$$=-37i$$

d $\frac{z_1z_3}{z_2^*}=\frac{z_1z_2z_3}{|z_2|^2}=\frac{(1+i)(3-2i)(-2+3i)}{3^2+(-2)^2}$

$$=\frac{(1+i)(13i)}{13}=i(1+i)=i-1$$

e $\frac{2z_1-4z_2^*}{z_3z_2^*}=\frac{2(1+i)-4(3+2i)}{(-2+3i)(3+2i)}$

$$= \frac{-10-6i}{-12+5i} = \frac{10+6i}{12-5i} = \frac{(10+6i)(12+5i)}{12^2+(-5)^2}$$

$$= \frac{90+122i}{169} = \frac{90}{169} + \frac{122i}{169}$$

2 a $\frac{1+2i}{i} = 2-i$ so $\mathrm{Re}(z) = 2$ and $\mathrm{Im}(z) = -1$

b $\frac{1}{i} + \frac{2i}{1-i} = -i + \frac{2i(1+i)}{2} = -i + i(1+i) = -1$

so $\mathrm{Re}(z) = -1$ and $\mathrm{Im}(z) = 0$

c $\frac{1+2i}{1-2i} - \frac{1-2i}{1+2i} = \frac{(1+2i)^2-(1-2i)^2}{(1-2i)(1+2i)}$

$$= \frac{(1+2i+1-2i)(1+2i-1+2i)}{5}$$

$$= \frac{8i}{5}$$

so $\mathrm{Re}(z) = 0$ and $\mathrm{Im}(z) = \frac{8}{5}$

3 a $(1+3i)(a+bi) = a-3b+(3a+b)i = 5+5i$

so $a-3b = 5$ and $3a+b = 5$

$\therefore 3a-9b = 15$

$\therefore 10b = -10 \Rightarrow b = -1$

$\Rightarrow a = 5+3b \Rightarrow a = 2$

b $\frac{a+bi}{1+2i} = -3+i$

$\Rightarrow a+bi = (-3+i)(1+2i) = -5-5i$

so $a = -5$ and $b = -5$

4 a $2(z+i) = 3i(z-1)$

$\Rightarrow z(2-3i) = -2i-3i = -5i$

$\therefore z = \frac{-5i}{2-3i} = \frac{-5i(2+3i)}{2^2+(-3)^2}$

$= \frac{-10i+15}{13} = \frac{15}{13} - \frac{10}{13}i$

b □ $\frac{z-2}{1-2i} = \frac{z-i}{2+i}$

$(z-2)(2+i) = (z-i)(1-2i)$

$\Rightarrow z(2+i-1+2i) = 2(2+i) - i(1-2i)$

$\Rightarrow (1+3i)z = 2+i$

$\Rightarrow z = \frac{2+i}{1+3i} = \frac{(2+i)(1-3i)}{1^2+3^2}$

$= \frac{5-5i}{10} = \frac{1-i}{2} = \frac{1}{2} - \frac{i}{2}$

c $(z+2i)(2+i)=(z-1)(1-i)$

$\therefore z(2+i-1+i)=-2i(2+i)-(1-i)$

$\Rightarrow z(1+2i)=1-3i$

$\Rightarrow z=\dfrac{1-3i}{1+2i}=\dfrac{(1-3i)(1-2i)}{1^2+2^2}=\dfrac{-5-5i}{5}=-1-i$

d $\dfrac{z+1+i}{1-4i}=\dfrac{z-3i+2}{2i+5}$

$\therefore (z+1+i)(2i+5)=(z-3i+2)(1-4i)$

$\Rightarrow z(2i+5-1+4i)=-(i+1)(2i+5)+(-3i+2)(1-4i)$

$\Rightarrow (6i+4)z=-3-7i-10-11i=-13-18i$

$\therefore z=-\dfrac{13+18i}{4+6i}=-\dfrac{(13+18i)(4-6i)}{4^2+6^2}$

$=-\dfrac{52+108-6i}{52}$

$=-\dfrac{40}{13}+\dfrac{3}{26}i$

5 **a** $\dfrac{a+bi}{2+i}=k\in\mathbb{R}\ \Rightarrow a+bi=2k+ki$

Comparing imaginary parts $\Rightarrow b=k$

Comparing real parts $\Rightarrow a=2k=2b$

so $a=2b$

b $\dfrac{1-i}{a-bi}=ki$ where $k\in\mathbb{R}$

$\therefore b+ai=\dfrac{1}{k}-\dfrac{1}{k}i$

Comparing real parts $\Rightarrow b=\dfrac{1}{k}$

Comparing imaginary parts $\Rightarrow a=-\dfrac{1}{k}=-b$

so $a=-b$

6 **a** $|z|+z=1$

$\Rightarrow \sqrt{x^2+y^2}+x+iy=1$

Comparing imaginary parts,

$y=0$

$\therefore \sqrt{x^2}+x-1=0$

$\Rightarrow |x|+x-1=0$

$x<0$ yields no solution $-x+x-1=0\Rightarrow$ false statement

for $x\geq 0$

$2x-1=0$

$\Rightarrow x=\dfrac{1}{2}$

b $|z|-z^*=i$

$\therefore \sqrt{x^2+y^2}-(x-iy)=i$

$\Rightarrow \sqrt{x^2+y^2}-x+iy=i$

Comparing real parts, $y=1$

$\therefore \sqrt{x^2+1}-x=0$

$\Rightarrow x^2+1=x^2 \Rightarrow 1=0$ false statement

Therefore, this has no solutions

c $z^2+z^*=2$

$\Rightarrow (x+iy)^2+(x-iy)=2$

$\Rightarrow x^2+2ixy-y^2+x-iy=2$

$\Rightarrow (x^2+x-y^2)+(2xy-y)i=2$

Comparing imaginary parts,

$2xy-y=0 \Rightarrow y(2x-1)=0$

so $y=0$ or $x=\frac{1}{2}$

If $x=\frac{1}{2}$,

$\left(\frac{1}{2}\right)^2+\frac{1}{2}-y^2=2$

$\Rightarrow y^2=-\frac{5}{4}$ which has no solutions

If $y=0$,

$x^2+x-2=0$

$(x-1)(x+2)=0$

so $x=1$ or $x=-2$

7 a $|z_1z_2|=|(x_1+iy_1)(x_2+iy_2)|=|(x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1)|$

$=\sqrt{(x_1x_2-y_1y_2)^2+(x_1y_2+x_2y_1)^2}$

$=\sqrt{x_1^2x_2^2+y_1^2y_2^2+x_1^2y_2^2+x_2^2y_1^2}$

$=\sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$

$=\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}=|z_1||z_2|$

b Follows from part a: replace z_2 with $\frac{1}{z_2}$

c $|z_1+z_2|^2=(z_1+z_2)(z_1+z_2)^*$

$=|z_1|^2+|z_2|^2+(z_1z_2^*+z_1^*z_2)$

$=|z_1|^2+|z_2|^2+2\mathrm{Re}(z_1z_2^*)$

$\le |z_1|^2+|z_2|^2+2|z_1z_2^*|$

$=|z_1|^2+|z_2|^2+2|z_1||z_2^*|$

$=|z_1|^2+|z_2|^2+2|z_1||z_2|$

$=(|z_1|+|z_2|)^2$

Since $|z_1+z_2|$ and $|z_1|+|z_2|$ are non-negative

$|z_1+z_2|\le|z_1|+|z_2|$

8 a $\left(z^*\right)^* = \left(x - iy\right)^* = x + iy = z$

b $\left(z_1 + z_2\right)^* = \left(\left(x_1 + x_2\right) + i\left(y_1 + y_2\right)\right)^*$

$= \left(x_1 + x_2\right) - i\left(y_1 + y_2\right)$
$= \left(x_1 - iy_1\right) + \left(x_2 - iy_2\right)$
$= z_1^* + z_2^*$

c $\left(z_1 z_2\right)^* = \left(\left(x_1 + iy_1\right)\left(x_2 + iy_2\right)\right)^*$

$= \left(x_1x_2 - y_1y_2 + i\left(x_1y_2 + x_2y_1\right)\right)^*$
$= x_1x_2 - y_1y_2 - i\left(x_1y_2 + x_2y_1\right)$
$= \left(x_1 - iy_1\right)\left(x_2 - iy_2\right)$
$= z_1^* z_2^*$

d $\left(\frac{z_1}{z_2}\right)^* = \left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)^* = \left(\frac{\left(x_1 + iy_1\right)\left(x_2 - iy_2\right)}{x_2^2 + y_2^2}\right)^*$

$= \left(\frac{x_1x_2 + y_1y_2 + i\left(y_1x_2 - x_1y_2\right)}{x_2^2 + y_2^2}\right)^*$

$= \frac{x_1x_2 + y_1y_2 + i\left(x_1y_2 - x_2y_1\right)}{x_2^2 + y_2^2}$

and

$\frac{z_1^*}{z_2^*} = \frac{x_1 - iy_1}{x_2 - iy_2} = \frac{\left(x_1 - iy_1\right)\left(x_2 + iy_2\right)}{x_2^2 + y_2^2}$

$= \frac{x_1x_2 + y_1y_2 + i\left(x_1y_2 - x_2y_1\right)}{x_2^2 + y_2^2}$

so $\left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*}$

e $|z| = \sqrt{x^2 + y^2} = \sqrt{x^2 + \left(-y\right)^2} = |z^*|$

Exercise 3H

1 a $i^7 + i^{17} + i^{27} + i^{37}$

$= i^4 i^3 + \left(i^4\right)^4 i + \left(i^4\right)^6 i^3 + \left(i^4\right)^9 i$
$= i^3 + i + i^3 + i = -i + i - i + i = 0$

b $i^{173} - i^{272} + i^{351} - i^{766}$

$= i^{172} i - i^{272} + i^{348} i^3 - i^{764} i^2$
$= \left(i^4\right)^{43} i - \left(i^4\right)^{68} + \left(i^4\right)^{87} i^3 - \left(i^4\right)^{191} i^2$
$= i - 1 + i^3 + 1$
$= i - 1 - i + i = 0$

c $\left(3 + i^{77}\right)\left(1 - 2i^{93}\right) = \left(3 + i^{76} i\right)\left(1 - 2i^{92} i\right)$

$$= \left(3+\left(i^4\right)^{19} i\right)\left(1-2\left(i^4\right)^{23} i\right)$$
$$= (3+i)(1-2i)$$
$$= 5-5i$$

d $\dfrac{3i^{2018}+2i^{2019}}{4i^{2020}-3i^{2021}} = \dfrac{3i^{2016}i^2+2i^{2016}i^3}{4i^{2020}-3i^{2020}i} = \dfrac{3\left(i^4\right)^{504} i^2+2\left(i^4\right)^{504} i^3}{4\left(i^4\right)^{505}-3\left(i^4\right)^{505} i}$

$$= \frac{-3-2i}{4-3i} = -\frac{(3+2i)(4+3i)}{(4-3i)(4+3i)} = -\frac{6+17i}{25} = -\frac{6}{25}-\frac{17}{25}i$$

e $i+i^2+i^3+i^4=0 \Rightarrow \sum_{k=1}^{2019} i^k = 0+0+\ldots 0+i+i^2+i^3=-1$

$i\times i^2\times i^3\times i^4=-1 \Rightarrow \prod_{k=1}^{2019} i^k = (-1)^{504}\times i\times i^2\times i^3=-1$

$\dfrac{-1}{-1}=1$

f $i+i^3=0 \Rightarrow \sum_{k=1}^{1010} i^{2k-1}=0$

$i\times i^3=1 \Rightarrow \prod_{k=1}^{1010} i^{2k}=1$

$\dfrac{0}{1}=0$

2 **a** $(3+2i)^3 = (3)^3+3(3)^2(2i)+3(3)(2i)^2+(2i)^3$

$$= 27+54i-36-8i$$
$$= -9+46i$$

b $(1-3i)^4 = 1+4(-3i)+6(-3i)^2+4(-3i)^3+(-3i)^4$

$$= 1-12i+6(-9)+4(27i)+81$$
$$= 28+96i$$

c $(1-2i)^4+(1+2i)^4$

$$= \left[1+4(-2i)+6(-2i)^2+4(-2i)^3+(-2i)^4\right]+\left[1+4(2i)+6(2i)^2+4(2i)^3+(2i)^4\right]$$
$$= \left[1-4(2i)+6(2i)^2-4(2i)^3+(2i)^4\right]+\left[1+4(2i)+6(2i)^2+4(2i)^3+(2i)^4\right]$$
$$= 2\left(1+6(2i)^2+(2i)^4\right)$$
$$= 2(1-24+16) = -14$$

d $(1+i)^5-(1-i)^5$

$$= \left(1+5i+10i^2+10i^3+5i^4+i^5\right)-\left(1-5i+10i^2-10i^3+5i^4-i^5\right)$$
$$= 2\left[5i+10i^3+i^5\right]$$
$$= 2\left[5i-10i+i\right]$$
$$= -8i$$

3 **a** $\sqrt{i} = a+bi$

$\Rightarrow i = (a + bi)^2 = a^2 - b^2 + 2abi$

Comparing real part:

$a^2 - b^2 = 0 \Rightarrow a = \pm b$

If $a = b,\ 2ab = 2a^2 = 1 \Rightarrow a = \pm\dfrac{1}{\sqrt{2}}$

If $a = -b,\ 2ab = -2a^2 = 1$ no solutions since $a \in \mathbb{R}$

$\therefore \dfrac{1+i}{\sqrt{2}}$ or $\dfrac{-1-i}{\sqrt{2}}$ $\left[\text{or} \pm\left(\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}i\right)\right]$

b $\sqrt{-i} = a + bi \Rightarrow -i = (a + bi)^2 = a^2 - b^2 + 2abi$

Comparing real parts, $a^2 - b^2 = 0 \Rightarrow a = \pm b$

Comparing imaginary parts:

$-1 = 2ab$

if $a = b \Rightarrow -1 = 2a^2 \Rightarrow a^2 = -\dfrac{1}{2} \Rightarrow$ no solution

Only a solution if $a = -b$

$\therefore -1 = -2a^2 \Rightarrow a = \pm\dfrac{1}{\sqrt{2}}$

$\therefore \dfrac{1-i}{\sqrt{2}}$ or $\dfrac{-1+i}{\sqrt{2}}$ $\left[\text{or} \pm\left(\dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{2}}{2}i\right)\right]$

c $\sqrt{-21+20i} = a + bi \Rightarrow -21 + 20i = (a + bi)^2 = a^2 - b^2 + 2abi$

$\therefore a^2 - b^2 = -21$ and $ab = 10 \Rightarrow a = \dfrac{10}{b}$

$\therefore \dfrac{100}{b^2} - b^2 = -21$

$\Rightarrow b^4 - 21b^2 - 100 = 0$

$\Rightarrow (b^2 - 25)(b^2 + 4) = 0$

$b \in \mathbb{R} \Rightarrow b = \pm 5$

$\Rightarrow a = \dfrac{10}{\pm 5} = \pm 2$

$\therefore \pm(2 + 5i)$

d $\sqrt{\dfrac{5}{36} - \dfrac{i}{3}} = a + bi$

$$\frac{5}{36} - \frac{i}{3} = (a+bi)^2 = a^2 - b^2 + 2abi$$

$$\Rightarrow a^2 - b^2 = \frac{5}{36} \text{ and } 2ab = -\frac{1}{3}$$

$$\therefore a = -\frac{1}{6b}$$

$$\therefore \frac{1}{36b^2} - b^2 = \frac{5}{36}$$

$$\Rightarrow 1 - 36b^4 = 5b^2$$

$$\Rightarrow 36b^4 + 5b^2 - 1 = 0$$

$$\Rightarrow (9b^2 - 1)(4b^2 + 1) = 0$$

$$b \in \mathbb{R} \Rightarrow b = \pm\frac{1}{3} \Rightarrow a = \mp\frac{1}{2}$$

$$\therefore \pm\left(\frac{1}{2} - \frac{1}{3}i\right)$$

4 a The statement follows repeated application of the result

given in Exercise 3G Question 7a. Can be (quite trivially) proven formally using this property via induction (in a similar fashion to the below)

b $P(n)$: $(z^*)^n = (z^n)^*$, $n \in \mathbb{Z}^+$

$P(1)$ is true
Assume $P(k)$ is true for some $k \in \mathbb{Z}^+$
Then,

$(z^*)^{k+1} = (z^*)^k z^* = (z^k)^* z^*$

$= (z^k z)^*$ using Exercise 3G Question 8c

$= (z^{k+1})^*$

so $P(k) \Rightarrow P(k+1)$

Therefore it has been shown that $P(1)$ is true and that if $P(k)$ is true for some $k \in \mathbb{Z}^+$ then so is $P(k+1)$. Thus, the statement is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction

5 a $(1+i)^2 = 2i$

$(1+i)^3 = 1 + 3i - 3 - i = -2 + 2i$

$(1+i)^4 = (1+i)(-2+2i) = -4$

$(1+i)^5 = -4(1+i)$ which is a multiple of $(1+i)$

so it is clear that whenever $n = 4k$ $(k \in \mathbb{Z}^+)$, $(1+i)^n$ is real

b It immediately follows from above that when $n = 4k+2$ $(k = 0, 1, 2\ldots)$ $(1+i)^2$ is purely imaginary

6 a $(1-i)^{2n} = \left((1-i)^2\right)^n = (-2i)^n$

b $(1-i)^{2n+1} = (1-i)(1-i)^{2n} = (1-i)(-2i)^n$ using part a

Exercise 3I

1 **a** $q(x) = 2x^2 - 3x + 1$

b $q(x) = 3x^3 + x^2 + 3$

c $q(x) = x^4 - x^2 - 2$

2 **a** $q(x) = 3x^2 - 3x - 2, r = -3$

b $q(x) = 2x^2 - 5x + 5, r(x) = 6x - 15$

c $q(x) = x^2 + x, r(x) = -x^2 - x + 1$

Exercise 3J

1 **a** $q(x) = x^2 + 4x + 5, r = 11$

b $q(x) = 2x^2 - 3x - 1, r = 1$

c $q(x) = 2x^3 + 2x^2 - x + 3, r = 1$

d $q(x) = 3x^4 + 2x^3 - 2x^2 - x + 13, r = -81$

2 **a** $(x+1)(x-3)\overline{\big)2x^4 - x^3 - 32x^2 + 31x + 60}$ with quotient $2x^2 + 3x - 20$

b $2x^2 + 3x - 20 = (2x-5)(x+4)$

$2x^4 - x^3 - 32x^2 + 31x + 60 = (x+1)(x-3)(2x-5)(x+4)$

Exercise 3K

1 **a** $q(x) = x^2 - x + 3, r = 1$

b $q(x) = 3x^2 + x + 1, r = 1$

c $q(x) = x^3 + 2x^2 - 2x + 1, r = 4$

d $q(x) = x^4 - 2x^3 + x^2 - x + 3, r = -1$

2 $f(x) = (x^2+2)(2x^2 - 3x + 1) + x - 3$

$= 2x^4 - 3x^3 + x^2 + 4x^2 - 6x + 2 + x - 3$
$= 2x^4 - 3x^3 + 5x^2 - 5x - 1$

3 By factor theorem,

$f(-2) = 6(-2)^5 + 17(-2)^4 - 20(-2)^3 - 35(-2)^2 + 44(-2) + a$
$= 12 + a = 0 \Rightarrow a = -12$

Horner's algorithm

	6	+17	−20	−35	+44	+a
−2	6	5	−30	25	−6	12 + a

$12 + a = 0 \Rightarrow a = -12$

4 By factor theorem,

$f\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^4 + 5\left(-\frac{1}{2}\right)^3 - 4\left(-\frac{1}{2}\right)^2 + b\left(-\frac{1}{2}\right) + 1$
$= -\frac{1}{2} - \frac{b}{2} = 0 \Rightarrow b = -1$

Horner's algorithm

	2	5	−4	−b	1
$-\frac{1}{2}$	2	4	−6	3 + b	$-\frac{1}{2}-\frac{1}{2}b$

$-\frac{1}{2}-\frac{1}{2}b=0 \Rightarrow b=-1$

5 By factor theorem,

$f(1)=1+5+5+a+b=11+a+b=0 \Rightarrow a+b=-11$

$f(-2)=(-2)^4+5(-2)^3+5(-2)^2+a(-2)+b=0$

$\Rightarrow -2a+b=4$

Eliminating b,

$3a=-15 \Rightarrow a=-5$ and therefore $b=-6$

Horner's algorithm

	1	5	5	a	b
1	1	6	11	11 + a	11 + a + b
−2	1	4	3	5 + a	

$11+a+b=0$

$5+a=0 \Rightarrow a=-5 \Rightarrow b=-6$

6 By factor theorem,

$$f\left(\frac{1}{2}\right)=\frac{6}{2^5}+\frac{13}{2^4}-\frac{30}{2^3}-\frac{45}{4}+\frac{a}{2}+b=-14+\frac{a}{2}+b=0$$

$\Rightarrow a+2b=28$

By polynomial remainder theorem,

$f(1)=6+13-30-45+a+b=-40 \Rightarrow a+b=16$

Eliminating a,

$b=12 \Rightarrow a=4$

7 $f(x)=(x+2)g(x)+4$ for some polynomial $g(x)$

$f(-2)=4; f(5)=-3$

$$f(x)=(x^2-3x-10)\bullet g(x)+\overbrace{ax+b}^{r(x)}$$

$$\left.\begin{aligned}4&=a(-2)+b\\-3&=a(5)+b\end{aligned}\right\}\text{Elimination}$$

$7=-7a \Rightarrow a=-1$

$4=2+b \Rightarrow b=2$

$r(x)=-x+2$

8 $f(-1)=(-1)^{2019}+(-1)^{2018}+\ldots+1=0$

so in fact $(x+1)$ is factor of $f(x)$

9 $f(x)=(x+2)^{2n}+(x+3)^n-1$

$x^2+5x+6=(x+3)(x+2)$ so $f(x)$ is divisible by

x^2+5x+6 if and only if it is divisible by both $(x+3)$ and $(x+2)$.

$f(-2)=(1)^n-1=0$ so $f(x)$ is divisible by $(x+2)$

$f(-3)=(-1)^{2n}-1=1^{2n}-1=0$ so $f(x)$ is divisible by $(x+3)$

Thus $f(x)$ is divisible by x^2+5x+6

10 By polynomial remainder theorem,

$f(x)=\left(x-\frac{b}{a}\right)q(x)+f\left(\frac{b}{a}\right)$ for some polynomial $q(x)$

$\Rightarrow af(x)=(ax-b)q(x)+af\left(\frac{b}{a}\right)$

i.e. the function $af(x)$ leaves remainder $af\left(\frac{b}{a}\right)$ when divided by $(ax-b)$.

Thus $f(x)$ leaves a remainder of $f\left(\frac{b}{a}\right)$ when divided by $(ax-b)$.

Exercise 3L

1 a $f(x)=x(x-2)(x-7)=x^3-9x^2+14x$

b $f(x)=(x+3)(x+2)(x-1)(x-3)$

$=(x^2+5x+6)(x^2-4x+3)$

$=x^4+x^3-11x^2-9x+18$

c $f(x)=2(x+1)\left(x+\frac{1}{2}\right)(x-2)(x-5)$

$=(x+1)(2x+1)(x^2-7x+10)$

$=(2x^2+3x+1)(x^2-7x+10)$

$=2x^4-11x^3+23x+10$

2 a $f(x)=(x^2-2)(x-2)=x^3-2x^2-2x+4$

b $f(x)=2(x+1)\left(x-\frac{1}{2}\right)(x^3-3)$

$=(x+1)(2x-1)(x^3-3)$

$=(2x^2+x-1)(x^3-3)$

$=2x^5+x^4-x^3-6x^2-3x+3$

c $f(x)=\left(x-\left(1-\sqrt{3}\right)\right)\left(x-\left(1+\sqrt{3}\right)\right)(x^3-2)$

$=(x^2-2x-2)(x^3-2)$

$=x^5-2x^4-2x^3-2x^2+4x+4$

3 a $f(x)=(x-1)(x^2-2x+2)$

b $f(x)=3x^3-x^2+2x+6=(x+1)(3x^2-4x+6)$

c $f(x) = 2x^4 - 5x^3 + 11x^2 - 3x - 5$

$= (x-1)(2x^3 - 3x^2 + 8x + 5)$
$= (x-1)(2x+1)(x^2 - 2x + 5)$

4 **a** $\square\ f(x) = (x+2)^2(ax+b) = (x^2 + 4x + 4)(ax+b)$

Now, long division or synthetic division can be used, though it is easier to note the coefficient of x^3 is 3 $\Rightarrow a = 3$
and $4b = -20 \Rightarrow b = -5$
$\therefore f(x) = (x+2)^2(3x-5)$

b $f(x) = (3x-2)^2(ax+b) = (9x^2 - 12x + 4)(ax+b)$

Now, long division or synthetic division can be used, though it is easier to note the coefficient of x^3 is 9 $\Rightarrow a = 1$
and $4b = 16 \Rightarrow b = 4$
$\therefore f(x) = (3x-2)^2(x+4)$

c $f(x) = (x-1)^2(ax^2 + bx + c) = (x^2 - 2x + 1)(ax^2 + bx + c)$

Now, long division or synthetic division can be used, though it is easier to note the coefficient of x^4 is 1 $\Rightarrow a = 1$
and $c = -4$ and the coefficient of x is
$8 = -2c + b \Rightarrow b = 8 + 2c = 0$
$\therefore f(x) = (x^2 - 2x + 1)(x^2 - 4) = (x-1)^2(x-2)(x+2)$

d $f(x) = (2x+1)^3(ax+b) = (8x^3 + 12x^2 + 6x + 1)(ax+b)$

Now, long division or synthetic division may be used, but it is easier to note the coefficient of x^4 is 8 $\Rightarrow a = 1$
and $b = 1$
$\therefore f(x) = (2x+1)^3(x+1)$

e $f(x) = (x-1)^4(ax+b) = (x^4 - 4x^3 + 6x^2 - 4x + 1)(ax+b)$

Now, long division or synthetic division may be used, but it is easier to note the coefficient of x^5 is 5 $\Rightarrow a = 5$
and $b = 7$
$\therefore f(x) = (x-1)^4(5x+7)$

5 **a** If $z = 2i$ is a root, then so is $z = -2i$

$\therefore f(z) = (z+2i)(z-2i)(az+b) = (z^2+4)(az+b)$
Now, long division or synthetic division can be used but it is easier to note that the coefficient of z^3 is $1 \Rightarrow a = 1$ and $4b = -8 \Rightarrow b = -2$
$\therefore f(z) = (z+2i)(z-2i)(z-2)$
So the remaining roots are $-2i$ and 2

b If $z = 3 - 2i$ is a zero then so is $z = 3 + 2i$

$\therefore f(z) = (z-(3-2i))(z-(3+2i))(az+b)$

$= (z^2 - 6z + 13)(az+b)$

Now, long division or synthetic division can be used but it is easier to note that the coefficient of z^3 is $2 \Rightarrow a = 2$

and $13b = -13 \Rightarrow b = -1$

$\therefore f(z) = (z-(3-2i))(z-(3+2i))(2z-1)$

So the remaining roots are $3+2i$ and $\frac{1}{2}$

c If $z = \frac{1}{2} - \frac{\sqrt{3}}{2}i$ is a root, then so is $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$\therefore f(z) = \left(z - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)\left(z - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)(az+b)$

$= (z^2 + z - 1)(az+b)$

Now, long division or synthetic division can be used but it is easier to note that the coefficient of z^3 is $3 \Rightarrow a = 3$

and $b = 7$

$\therefore f(z) = \left(z - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)\left(z - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)(3z+7)$

So the remaining roots are $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $-\frac{7}{3}$

d If $z = -i$ is a zero then so is $z = i$

$\therefore f(z) = (z-i)(z+i)(az^2 + bz + c)$

$= (z^2+1)(az^2+bz+c)$

Now, long division or synthetic division can be used, but it is easier to note that the coefficient of z^4 is $1 \Rightarrow a = 1$

and $c = 5$ and the coefficient of z is

$-2 = b$

$\therefore f(z) = (z^2+1)(z^2-2z+5)$

$\Rightarrow z^2 - 2z + 5 = 0$

$\Rightarrow (z-1)^2 = -4$

$\Rightarrow z = 1 \pm 2i$

So the remaining roots are $z = i$ and $z = 1 \pm 2i$

e If $z = -2 - i$ is a zero then so is $z = -2 + i$

$\therefore f(z) = (z-(-2-i))(z-(-2+i))(az^2+bz+c)$

$= (z^2+4z+5)(az^2+bz+c)$

Now, long division or synthetic division can be used, but it is easier to note that the coefficient of z^4 is $1 \Rightarrow a = 1$

and $5c = 10 \Rightarrow c = 2$

and the coefficient of z is $13 = 4c + 5b \Rightarrow b = \frac{13-8}{5} = 1$

$f(z) = (z^2+4z+5)(z^2+z+2)$

so the remaining zeros satisfy

$z^2+z+2 = \left(z+\frac{1}{2}\right)^2 = -\frac{7}{4}$

$\Rightarrow z = -\frac{1}{2} \pm \frac{\sqrt{7}}{2}i$

So the remaining roots are $-2+i$ and $-\frac{1}{2} \pm \frac{\sqrt{7}}{2}i$

f If $z = -\frac{1}{3} + \frac{\sqrt{2}}{3}i$ then so is $z = -\frac{1}{3} - \frac{\sqrt{2}}{3}i$

$\therefore f(z) = \left(z - \left(-\frac{1}{3} + \frac{\sqrt{2}}{3}i\right)\right)\left(z - \left(-\frac{1}{3} - \frac{\sqrt{2}}{3}i\right)\right)(az^2+bz+c)$

$= \left(z^2 + \frac{2}{3}z + \frac{1}{3}\right)(az^2+bz+c)$

Now, long division or synthetic division may be used, but it is easier to note that the coefficient of z^4 is $3 \Rightarrow a = 3$

and $\frac{c}{3} = 6 \Rightarrow c = 18$

and the coefficient of z is $12 = \frac{2c}{3} + \frac{b}{3} \Rightarrow b = 36 - 2(18) = 0$

$\therefore f(z) = \left(z^2 + \frac{2}{3}z + \frac{1}{3}\right)(3z^2+18)$

$\Rightarrow 3z^2 + 18 = 0 \Rightarrow z = \pm i\sqrt{6}$

so the remaining roots are $z = \frac{-1-\sqrt{2}i}{3}$ and $z = \pm\sqrt{6}i$

6 a $f(2) = 8 + 4a + 2 - 2 = 0 \Rightarrow a = -2$

$\therefore f(x) = x^3 - 2x^2 + x - 2 = (x-2)(x^2+1)$

roots of $x^2+1=0$ can be found by long division, synthetic division, or inspection so the remaining roots are $x = \pm i$

b $f(-5) = -250 + 250 - 5a + 15 = 0 \Rightarrow a = 3$

$f(x) = 2x^3 + 10x^2 + 3x + 15$

$= (x+5)(2x^2+3)$

roots of $2x^2+3=0$ can be found by long division, synthetic division, or inspection

so the remaining roots are $x = \pm\sqrt{\frac{3}{2}}i$

c $f(x) = x^4 - 2x^3 + ax^2 + bx + 85$

If $x = \sqrt{5}i$ is a root then so is $x = -\sqrt{5}i$

$\therefore f\left(\sqrt{5}i\right) = 25 + 10\sqrt{5}i - 5a + \sqrt{5}bi + 85 = 0$

Comparing real parts, $25 - 5a + 85 = 0 \Rightarrow a = 22$

Comparing imaginary parts, $10\sqrt{5} + \sqrt{5}b = 0 \Rightarrow b = -10$

$\therefore f(x) = x^4 - 2x^3 + 22x^2 - 10x + 85$

$= \left(x - \sqrt{5}i\right)\left(x + \sqrt{5}i\right)\left(cx^2 + dx + e\right) = \left(x^2 + 5\right)\left(cx^2 + dx + e\right)$

Now, long division or synthetic division can be used, but it is easier to note that the coefficient of x^4 is $1 = c$

and $5e = 85 \Rightarrow e = 17$

and the coefficient of x is $5d = -10 \Rightarrow d = -2$

$\therefore x^2 - 2x + 17 = (x-1)^2 + 16 = 0 \Rightarrow x = 1 \pm 4i$

So the remaining zeros are

$x = -\sqrt{5}i,\ x = 1 + 4i,\ x = 1 - 4i$

d If $x = 1 - 2i$ is a root then so is $x = 1 + 2i$

$f(1-2i) = 2 + a + b - (14 + 2a)i = 0$

Comparing real and imaginary parts,

$\therefore a = -7 \Rightarrow b = 5$

$\therefore f(x) = \left(x - (1-2i)\right)\left(x - (1+2i)\right)\left(cx^2 + dx + e\right)$

$= \left(x^2 - 2x + 5\right)\left(cx^2 + dx + e\right)$

$= 3x^4 - 7x^3 + 18x^2 - 7x + 5$

Now, long division or synthetic division can be used, but it is easier to note that the coefficient of x^4 is $3 \Rightarrow c = 3$

and $5e = 5 \Rightarrow e = 1$

and the coefficient of x is $5d - 2e = -7 \Rightarrow d = -1$

$\therefore 3x^2 - x + 1 = 0 \Rightarrow x = \dfrac{1 \pm i\sqrt{11}}{6}$

so the remaining roots are

$x = 1 + 2i$ and $x = \dfrac{1 \pm i\sqrt{11}}{6}$

Exercise 3M

1 a $x_1 + x_2 + x_3 + x_4 = -\dfrac{-3}{1} = 3$

$x_1x_2x_3x_4 = \dfrac{4}{1} = 4$

b $\displaystyle\sum_{i=1}^{6} x_i = -\frac{-6}{2} = 3$

$\displaystyle\prod_{i=1}^{6} x_i = \frac{0}{2} = 0$

c $\displaystyle\sum_{i=0}^{17} x_1 = -\frac{0}{23} = 0$

$$\prod_{i=1}^{17} x_i = (-1)^{17} - \frac{46}{23} = 2$$

d $$\sum_{i=1}^{2020} x_i = -\frac{-4}{3} = \frac{4}{3}$$

$$\prod_{i=1}^{2020} x_i = (-1)^{2020} \times -\frac{8}{3} = -\frac{8}{3}$$

2 **a** $\square$ $x_1 + x_2 + x_3 = -\frac{-2}{4} = \frac{1}{2}$

b $x_1 x_2 x_3 = (-1)^3 \frac{-17}{4} = \frac{17}{4}$

c $10x_1 + 10x_2 + 10x_3 = 10(x_1 + x_2 + x_3) = 10\left(-\frac{-2}{4}\right) = 5$

d $$\frac{3}{x_1x_2} + \frac{3}{x_1x_3} + \frac{3}{x_2x_3} = \frac{3(x_1 + x_2 + x_3)}{x_1x_2x_3} = \frac{3\left(\frac{1}{2}\right)}{\frac{17}{4}} = \frac{6}{17}$$

3 **a** $x_1 + x_2 + x_3 + x_4 = -\frac{2}{6} = -\frac{1}{3}$

b $x_1x_2x_3x_4 = (-1)^4 \frac{-3}{6} = -\frac{1}{2}$

c $3x_1 + 3x_2 + 3x_3 + 3x_4 = 3(x_1 + x_2 + x_3 + x_4) = 3\left(-\frac{1}{3}\right) = -1$

d $$\frac{6}{x_1x_2x_3} + \frac{6}{x_1x_2x_4} + \frac{6}{x_1x_3x_4} + \frac{6}{x_2x_3x_4} = \frac{6(x_1 + x_2 + x_3 + x_4)}{x_1x_2x_3x_4}$$

$$= \frac{6\left(-\frac{1}{3}\right)}{-\frac{1}{2}} = 4$$

Exercise 3N

1 **a** $x^3 + x^2 - 4x - 4 = (x+1)(x^2 - 4) = (x+1)(x+2)(x-2) = 0$

$\therefore x = -2,\ x = -1,\ x = 2$

b $x^3 + 2x^2 - 9x - 18 = (x+2)(x^2 - 9) = (x+2)(x+3)(x-3) = 0$

$\therefore x = -3,\ x = -2,\ x = 3$

c $x^3 - 3x^2 + 3x - 2 = (x-2)(x^2 - x + 1) = 0$

$x^2 - x + 1 = 0$ exhibits no real solution since $\Delta = 1^2 - 4 = -3 < 0$
$\therefore x = 2$

d $x^4 + x^3 - 3x^2 - x + 2 = (x+1)(x^3 - 3x + 2) = (x-1)^2(x+2)(x+1) = 0$

$\therefore x = -2,\ x = -1,\ x = 1$

2 a $(-2)^3 + (-2)^2 + a(-2) - 4 = 0 \Rightarrow a = -4$

b $x^3 + x^2 - 4x - 4 = (x+2)(x^2 - x - 2) = (x+2)(x-2)(x+1) = 0$

So the remaining roots are $x = 2$ and $x = -1$

3 a $2x^3 + ax^2 + bx + 9 = (x-3)^2(cx+d) = (x^2 - 6x + 9)(cx+d)$

Coefficient of x^3 : $2 = c$
Units: $9 = 9d \Rightarrow d = 1$
$\therefore 2x^3 + ax^2 + bx + 9 = (x^2 - 6x + 9)(2x+1)$
$= 2x^3 - 11x^2 + 12x + 9$
$\therefore a = -11,\ b = 12$

b From the factorised form, deduce that the remaining root is $x = -\dfrac{1}{2}$

4 a Let the three roots be $-x_1, x_1$ and x_2

$(x - x_1)(x + x_1)(x - x_2) = 0$
$(x^2 - x_1^2)(x - x_2) = 0$
$x^3 - x_2x^2 - x_1^2x + x_1^2x_2 = 0$
$a = -x_2, b = -x_1^2, c = x_1^2x_2$
$\Rightarrow ab = c$

b From part a, $x_2 = -a$

Exercise 3O

1 a $x^3 + x^2 - 4x - 4 < 0$

$\Rightarrow -1 < x < 2,\ x < -2$

b $x^3 + 2x^2 - 9x - 18 > 0$

$\Rightarrow x > 3,\quad -3 < x < -2$

c $x^3 - 3x^2 + 3x - 2 \geq 0$

$\Rightarrow x \leq 2$

d $4x^3 + 8x^2 + x - 3 \geq 0$

$\Rightarrow x \geq \frac{1}{2}, \quad -\frac{3}{2} \leq x \leq -1$

e $3x^3 + 4x^2 + 7x + 2 > 0$

$\Rightarrow x > -\frac{1}{3}$

f $12x^3 - 16x^2 - 81x - 35 < 0$

$\Rightarrow x < -\frac{5}{3}, \quad -\frac{1}{2} < x < \frac{7}{2}$

g $x^4 + x^3 - 3x^2 - x + 2 \geq 0$

$\Rightarrow x \leq -2, \ x \geq -1$

h $2x^4 - x^3 + x^2 - x - 1 < 0$

$-\frac{1}{2} < x < 1$

2 $f(x) \leq g(x)$

$\Rightarrow 2x^3 + 3x^2 - 2x - 5 \leq 0$
$\Rightarrow x < 1.17227...$

3 **a** Rearrange to give $x^3 > -x - 2$

Plot $y = x^3$ and $y = -2 - x$ and using a GDC or by inspection deduce
$x > -1$

b Rearrange to give $x^2 + 1 \geq 2x^3$

Plot $y = 2x^3$ and $y = x^2 + 1$ and using a GDC or by inspection deduce
$x \leq 1$

c Rearrange to give $x^4 \leq 2 - x^2$

Plot $y = x^4$ and $y = 2 - x^2$ and using a GDC or by inspection deduce
$-1 \leq x \leq 1$

4 **a** $x^5 - 4x^3 + 2x + 1 \geq 0$

$\Rightarrow x > 1.78897..., \quad -1.8947... < x < 1$

b $x^{13} + 4 < 3x^8 + 5x$

$0.746571... < x < 1.27299..., \quad x < -1.09526$

c $x^{15} + 2x^{14} + 5x^8 > 4x^2 - 1$

$-2.06403... < x < -0.888753..., \ -0.505311... < x < 0.505331..., \ x > 0.868507...$

Exercise 3P

1 **a** Multiplying the second equation by 2,

$14x + 6y = 10$
Therefore, subtracting this from the first equation,
$(m - 14)x = -12$
$\Rightarrow$ No solution if $m = 14$

b Multiply the first equation by $m: m(m-1)x + 2my = -m$
Multiply the second equation by 2: $6x + 2my = 22$
Subtracting these equations, $[m(m-1) - 6]x = -m - 22$
$\therefore$ No solution if $m(m-1) - 6 = m^2 - m - 6 = (m-3)(m+2) = 0$
$\Rightarrow m = 3$ or $m = -2$

2 **a** Multiplying the second equation by 2,

$6x + 2y = 8$
so by comparison with the first equation, there are infinitely many solutions if $p = 2$

b Equating ratios of coefficients and constants:

$\therefore \dfrac{p-4}{p} = \dfrac{p-1}{-1} = \dfrac{-2}{p}$
$\Rightarrow \dfrac{p-4}{p} = \dfrac{p(1-p)}{p} = \dfrac{-2}{p}$

$$\left.\begin{array}{l}(1):(p-4) = p(1-p)\\(2):(1-p)p = -2\\(3):(p-4) = -2\end{array}\right\}\text{Have only one common solution, } p = 2$$

$\Rightarrow p = 2$ gives infinitely many solutions.

3 **a** Multiply the first equation by 3

$6x - 3sy = 3$
Multiply the second equation by 2,
$6x - 2y = 4$
Subtracting these equations,
$(2 - 3s)y = -1$
Therefore for a unique solution we need
$2 - 3s \neq 0 \Rightarrow s \neq \dfrac{3}{2}$
Then, $y = -\dfrac{1}{2-3s} = \dfrac{1}{3s-2}$
and accordingly
$x = \dfrac{1}{2}(1 + sy) = \dfrac{1}{2}\left(1 + \dfrac{s}{3s-2}\right) = \dfrac{1}{2}\left(\dfrac{4s-2}{3s-2}\right) = \dfrac{2s-1}{3s-2}$

b Rearranging the first equation,

$y = s - (s+2)x$

Substituting this into the second equation

$(5-2s)x + s[s-(s+2)x] = 4$

$\Rightarrow [(5-2s) - s(s+2)]x = 4 - s^2$

$\Rightarrow [-s^2 - 4s + 5]x = 4 - s^2$

$\therefore s \neq 1,\ s \neq -5$

$\Rightarrow x = \dfrac{4-s^2}{-s^2-4s+5} = \dfrac{s^2-4}{s^2+4s-5}$

$\Rightarrow y = s - \dfrac{(s+2)(s^2-4)}{s^2+4s-5} = \dfrac{s^3+4s^2-5s-(s^3+2s^2-4s-8)}{s^2+4s-5}$

$= \dfrac{2s^2-s+8}{s^2+4s-5}$

4 a Adding the equations,

$(2a+2b)x = 2 \Rightarrow x = \dfrac{1}{a+b}$

Subtracting the equations,

$(2a-2b)y = 0 \Rightarrow y = 0$

b The system does not have a solution when $a+b=0$

Exercise 3Q

1 $a = 3, b = 1+i, c = 4-i, d = -(1-i), \mathrm{e} = 4+5i, f = 7$

$D = ad - bc = 3(i-1) - (4-i)(1+i) = -8$

$x_{num} = \mathrm{e}d - fb = -(4+5i)(1-i) - 7(1+i) = -16-8i$

$y_{num} = af - \mathrm{e}c = 3(7) - (4+5i)(4-i) = -16i$

$\begin{pmatrix} x \\ y \end{pmatrix} = -\dfrac{1}{8}\begin{pmatrix} -16-8i \\ -16i \end{pmatrix} = \begin{pmatrix} 2+i \\ 2i \end{pmatrix}$

2 $a = 1+4i, b = 3i, c = 3-5i, d = 5+4i, \mathrm{e} = 2+4i, f = 21-27i$

$D = ad - bc = (1+4i)(5+4i) - 3i(3-5i) = -26+15i$

$x_{num} = \mathrm{e}d - fb = (5+4i)(2+4i) - 3i(21-27i) = -87-35i$

$y_{num} = af - \mathrm{e}c = (1+4i)(21-27i) + (2+4i)(3-5i) = 103+55i$

$\begin{pmatrix} x \\ y \end{pmatrix} = \dfrac{1}{-26+15i}\begin{pmatrix} -87-35i \\ 103+55i \end{pmatrix}$

Exercise 3R

1 a Eliminating y from the first two equations:

$3x+(m-1)z=6 \Rightarrow x+\frac{m-1}{3}z=2$

and from the second two equations,

$5x+(2m-1)z=5 \Rightarrow x+\frac{2m-1}{5}z=1$

Eliminating x,

$\left(\frac{2m-1}{5}-\frac{m-1}{3}\right)z=-1$

$\Rightarrow \frac{6m-3-5m+5}{15}z=-1$

$\Rightarrow \frac{m+2}{15}z=-1$

$\therefore m \neq -2$

b Eliminating x from first two equations,

$(1-3m)y+(m-3)z=-5 \quad (1)$

and from the first and third:

$(2-m)y=2 \Rightarrow m \neq 2$

and substituting this into equation (1)

$\Rightarrow m \neq 3$

so $m \neq 2$ and $m \neq 3$

2 **a** Multiplying the first equation by 3,

$6x+9y-3z=3$

so comparing this with the second equation,

$k=6$

b $k=2$ gives infinitely many solutions as the the second equation is equal to 2 times the third equation

3 If $m=1$, then

$x+y+z=2$

and

$x+y+z=-3$

which is clearly not possible. If $m=0$, then

$2x+y+z=0$

and adding together equations (2) and (3) gives

$(x+z)+(x+y)=1+(-3)$

$2x+y+z=-2$

which is again clearly not possible. So it has one solution for all values of m other than $m=1$ and $m=0$.

4 **a** $9a-3b+c=1$

$4a-2b+c=-5$

$a+b+c=4$

Gaussian elimination on this system of equations gives

$a=\frac{9}{4},\ b=\frac{21}{4},\ c=-\frac{7}{2}$

b $a-b+c=1$

$a + b + c = -9$
$4a + 2b + c = 8$
Gaussian elimination on this system of equation gives
$a = \dfrac{22}{3},\ b = -5,\ c = -\dfrac{34}{3}$

Chapter review

1 $x(x - a - b) = 1 - ab$

$\Rightarrow x^2 - (a + b)x + ab - 1 = 0$
$\Delta = (a + b)^2 - 4(ab - 1)$
$= a^2 + 2ab + b^2 - 4ab + 4$
$= a^2 - 2ab + b^2 + 4$
$= (a - b)^2 + 4 > 0$
so there are two distinct real solutions for all $a, b \in \mathbb{R}$

2 **a** $(1 - i)^3 + (1 + i)^3 = \left(1 - 3i + 3(-i)^2 - i^3\right) + \left(1 + 3i + 3(-i)^2 + i^3\right)$

$= (1 - 3i - 3 + i) + (1 + 3i - 3 - i)$
$= (-2 - 2i) + (2i - 2)$ $\square$
$= -4$

b $\omega = \dfrac{i^{64} i^{63} \ldots i^{18}}{1 - 2i} = \dfrac{i^{64} i^{63} i^{62} i^{61} \ldots i^{19} i^{18}}{1 - 2i} = \dfrac{[1 \cdot i \cdot i^2 \cdot i^3]^{11} i^{19} i^{18}}{1 - 2i}$

$= \dfrac{(-1)^{11} i^3 i^2}{1 - 2i} = \dfrac{-(-i)(-1)}{1 - 2i} = \dfrac{-i}{1 - 2i} = -\dfrac{i(1 + 2i)}{1^2 + 2^2} = -\dfrac{i - 2}{5} = \dfrac{2}{5} - \dfrac{i}{5}$

c If ω is a root of $f(x)$ then so is ω^*

$\therefore f(x) = C(x + 4)\left(x - \dfrac{2 + i}{5}\right)\left(x - \dfrac{2 - i}{5}\right)$

$= C(x + 4)\left(x^2 - \dfrac{4}{5}x + \dfrac{1}{5}\right)$

$= \dfrac{C}{5}(x + 4)(5x^2 - 4x + 1)$

$= \dfrac{C}{5}(5x^3 + 16x^2 - 15x + 4)$
so set, for example, $C = 5$ to obtain integers coefficients
e.g. $f(x) = 5x^3 + 16x^2 - 15x + 4$

3 **a** $a(-2 - ay) + y = 2$

$(-a^2 + 1)y = 2 + 2a$
$\therefore -a^2 + 1 \neq 0 \Rightarrow a \neq \pm 1$
$x = -2 - \dfrac{2a}{1 - a} = \dfrac{-2 + 2a - 2a}{1 - a} = \dfrac{-2}{1 - a} = \dfrac{2}{a - 1}$
$y = \dfrac{2(1 + a)}{1 - a^2} = \dfrac{2(1 + a)}{(1 - a)(1 + a)} = \dfrac{2}{1 - a}$

b **i** $a = 1$ since then $x + y = 2$ and $x + y = -2$, which is not possible

ii $a = -1$

$y = x + 2$

4 **a** $\dfrac{(1+i)^{2019}}{(1-i)^{2017}} = \dfrac{(1+i)^{2019+2017}}{(1-i)^{2017}(1+i)^{2017}} = \dfrac{(1+i)^{4036}}{2^{2017}} = \dfrac{\left((1+i)^4\right)^{1009}}{2^{2017}}$

$\dfrac{(-4)^{1009}}{2^{2017}} = -\dfrac{2^{2018}}{2^{2017}} = -2$

b $\dfrac{(1+i)^{n+2}}{(1-i)^{n}} = \dfrac{(1+i)^{2n+2}}{2^n} = \dfrac{\left((1+i)^2\right)^{n+1}}{2^n} = \dfrac{(2i)^{n+1}}{2^n} = 2i^{n+1}$

$\therefore n + 1$ must be odd, so n must be even

5 $x^5 + 3x^4 - 2x^2 + 2x + 4 \geq x^4 + 3x^3 + 2x^2 + 3$

$x^5 + 2x^4 - 3x^3 - 4x^2 + 2x + 1 \geq 0$

Use your GDC:

$-2.45 \leq x \leq -1.26$,
$-0.339 \leq x \leq 0.715$,
$1.34 \leq x$ (3s.f.)

6 $\begin{cases} 3x - 2y + z = 1 \\ 6x + 8y - 3z = 6 \\ -12x + 4y - 7z = 4 \end{cases}$

Eliminating x from the first two equations,
$12y - 5z = 4 \quad (1)$
and from the second two equations,
$20y - 13z = 16 \quad (2)$
Three times (2) minus five times (1),
$-14z = 28 \Rightarrow z = -2$
$\Rightarrow y = -\dfrac{1}{2}$
$\Rightarrow x = \dfrac{2}{3}$

7 **a** 2

b $\square$ $(f(x))^2 = 9x^4 + 12x^3 - 26x^2 - 20x + 25$

$= (3x^2 + 2x - 5)^2 = \left((x-1)(3x+5)\right)^2$

$\therefore \alpha = 1,\ \beta = -\dfrac{5}{3}$

$\therefore \dfrac{1}{\alpha} + \dfrac{1}{\beta} = 1 - \dfrac{3}{5} = \dfrac{2}{5}$

8 $\begin{cases} z^* - i\omega = 3 \\ \omega^* + iz = 6 - i \end{cases} \Rightarrow \begin{cases} z + i\omega^* = 3 \\ -z + i\omega^* = 1 + 6i \end{cases}$

$\therefore 2i\omega^* = 4 + 6i$
$\Rightarrow 2i\left(a - bi\right) = 4 + 6i$
$\therefore a = 3,\ b = 2$ so $\omega = 3 + 2i$
$\Rightarrow z = 3 - i\left(3 - 2i\right) = 1 - 3i$

9 Using sum of a geometric sequence,

$$1 - z + z^2 = \frac{\left(-z\right)^3 - 1}{-z - 1} = 0 \Rightarrow z^3 = -1$$
$$\therefore z^{2019} = \left(z^3\right)^{673} = \left(-1\right)^{673} = -1$$

Exam-style questions

10 a $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (1 mark)

$x = \dfrac{6 \pm \sqrt{208}}{2}$ (1 mark)

$x = 3 \pm \sqrt{52}$ (1 mark)

$x = 3 \pm 2\sqrt{13}$ (1 mark)

b Using sketch or table

$3 - 2\sqrt{13} \le x \le 3 + 2\sqrt{13}$ (2 marks)

11 a $8x^2 + 6x - 5 = 0$

$\left(4x + 5\right)\left(2x - 1\right) = 0$ (2 marks)

$4x + 5 = 0 \Rightarrow x = -\dfrac{5}{4}$ (1 mark)

$2x - 1 = 0 \Rightarrow x = \dfrac{1}{2}$ (1 mark)

b $8x^2 + 6x - 5 - k = 0$

No real solutions $\Rightarrow b^2 - 4ac < 0$ (1 mark)

$36 - 4 \times 8 \times \left(-5 - k\right) < 0$ (1 mark)

$36 + 32(5 + k) < 0$

$5 + k < -\frac{36}{32}$

$k < -\frac{36}{32} - 5$

$k < -\frac{9}{8} - \frac{40}{8}$

$k < -\frac{49}{8}$ (1 mark)

12 Two real roots implies $b^2 - 4ac > 0$ (1 mark)

$\left(k\sqrt{3}\right)^2 - 4(3k)(3) > 0$ (1 mark)

$3k^2 - 36k > 0$

$k^2 - 12k > 0$

$k(k - 12) > 0$ (1 mark)

Critical values are $k = 0$ and $k = 12$ (1 mark)

Solution is $k < 0$ or $k > 12$ (2 marks)

13 a Using sketch of $y = (x + 4)(3 - x)$ (1 mark)

Correct sketch or table

Solution is $-4 < x < 3$ (2 marks)

b $2x^2 - 11x + 9 < 0$

$(2x - 9)(x - 1) < 0$ (1 mark)

Using sketch of $y = (2x - 9)(x - 1)$ (1 mark)

Correct sketch or table

Solution is $1 < x < \frac{9}{2}$ (2 marks)

c Comparing answers from **a** and **b** gives

$1 < x < 3$ (1 mark)

14 Let $w = a + b\mathrm{i}$ where $a, b \in \mathbb{R}$

$(a + b\mathrm{i})^2 = 77 - 36\mathrm{i}$ (1 mark)

$a^2 - b^2 + 2ab\mathrm{i} = 77 - 36\mathrm{i}$ (1 mark)

Equating reals: $a^2 - b^2 = 77$ (1) (1 mark)

Equating imaginary: $2ab = -36$ (2) (1 mark)

(2) gives $b = -\dfrac{18}{a}$

Substitute in (1): $a^2 - \left(-\dfrac{18}{a}\right)^2 = 77$ (1 mark)

$a^2 - \dfrac{324}{a^2} = 77$

$a^4 - 77a^2 - 324 = 0$

Attempting to factorise, or using the quadratic formula:

$\left(a^2 + 4\right)\left(a^2 - 81\right) = 0$ (1 mark)

Since $a \in \mathbb{R}$, $a^2 = 81$ (1 mark)

$a = \pm 9$

$a = 9 \Rightarrow b = -2$

$a = -9 \Rightarrow b = 2$

So $w = \pm(9 - 2\mathrm{i})$ (2 marks)

15 Let $p(x) = 2x^3 + ax^2 - 10x + b$

$p(1) = 0 \Rightarrow 2 + a - 10 + b = 0$ (2 marks)

$a + b = 8$ (1) (1 mark)

$p(-2) = 15 \Rightarrow -16 + 4a + 20 + b = 15$

$4a + b = 11$ (2) (2 marks)

Solving equations (1) and (2) simultaneously: (1 mark)

$a = 1$ (1 mark)

$b = 7$ (1 mark)

16 Since $3 - \mathrm{i}$ is a zero, its conjugate is also a zero, i.e. $(3 - \mathrm{i})^* = 3 + \mathrm{i}$ is a zero. (1 mark)

By the factor theorem,

$\left[z - (3 - i)\right]$ is a factor of f, and $\left[z - (3 + i)\right]$ is a factor of f. (1 mark)

Therefore $\left[z - (3 - i)\right]\left[z - (3 + i)\right] = z^2 - 6z + 10$ is also a factor of f. (2 marks)

Writing $z^4 - 8z^3 + 48z^2 - 176z + 260 = \left[z^2 - 6z + 10\right]\left[z^2 + kz + 26\right]$

Equating coefficients of z^2 gives $48 = 26 - 6k + 10$ (2 marks)

So $k = -2$ (1 mark)

$z^2 - 2z + 26 = 0$

$z = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 1 \times 26}}{2} = \frac{2 \pm \sqrt{-100}}{2} = \frac{2 \pm 10\text{i}}{2} = 1 \pm 5\text{i}$ (2 marks)

The zeros are therefore $(3 \pm \text{i})$ and $(1 \pm 5\text{i})$.

17 a Sum of roots $= -\left(\frac{-4}{5}\right) = \frac{4}{5}$ (2 marks)

Product of roots $= (-1)^4\left(-\frac{1}{5}\right) = -\frac{1}{5}$ (2 marks)

b Sum of roots $= -\left(\frac{-4}{5}\right) = \frac{4}{5}$ (2 marks)

Product of roots $= (-1)^5\left(\frac{10}{5}\right) = -2$ (2 marks)

18 Suppose $w = a + b\text{i}$ and $z = c + d\text{i}$ for $a, b, c, d \in \mathbb{R}$ (1 mark)

Then $w^* = a - b\text{i}$ and $z^* = c - d\text{i}$ (1 mark)

So $wz^* - zw^* = (a + b\text{i})(c - d\text{i}) - (c + d\text{i})(a - b\text{i})$

$= [ac + bd + \text{i}(bc - ad)] - [ac + bd + \text{i}(ad - bc)]$ (1 mark)

$= \text{i}(bc - ad) - \text{i}(ad - bc)$ (1 mark)

$= \text{i}(bc - ad) + \text{i}(bc - ad)$

$= 2(bc - ad)\text{i}$

which is purely imaginary (1 mark)

19 At $(-1, -5)$: $a - b + c = -5$ (1 mark)

At $(3, -1)$: $9a + 3b + c = -1$ (1 mark)

At $(10, -71)$: $100a + 10b + c = -71$ (1 mark)

Solving simultaneously using GDC: (1 mark)

$a = -1$ (1 mark)

$b = 3$ (1 mark)

$c = -1$ (1 mark)

4 Measuring change: differentiation

Skills check

1 $7x^{\frac{1}{2}}; 2x^{-3}; \frac{8}{5}x^{\frac{-2}{3}}$

2 Vertical asymptote: x = − 3

Horizontal asymptote: y = 0

y-intercept: $\left(0, \frac{2}{3}\right)$

3 $S_\infty = 5 \times \frac{1}{1-\frac{1}{2}} = 5 \times 2 = 10$

Exercise 4A

1 $\lim\limits_{x\to -2} \frac{x^2-4}{x+2} = \lim\limits_{x\to -2} \frac{(x-2)(x+2)}{x+2} = \lim\limits_{x\to -2}(x-2) = -4$

2 $\lim\limits_{x\to 3} \frac{x^2-9}{x-3} = \lim\limits_{x\to 3} \frac{(x-3)(x+3)}{x-3} = \lim\limits_{x\to 3}(x+3) = 6$

3 □ $\lim\limits_{x\to 2^-}(x-3) = -1$ whereas $\lim\limits_{x\to 2^+}(x+1) = 3$ so the limit does not exist

4 $\lim\limits_{x\to 1} \frac{x^3-1}{x-1} = \lim\limits_{x\to 1} \frac{(x-1)(x^2+x+1)}{x-1} = \lim\limits_{x\to 1}(x^2+x+1) = 3$

5 $\lim\limits_{x\to 1^-}(x+1) = 2$ and $\lim\limits_{x\to 1^+}(x^2+1) = 2$ so $\lim\limits_{x\to 1} f(x) = 2$

6 $\lim\limits_{x\to 6}\left((x-6)^{\frac{2}{3}}\right) = 0$

7 $\lim\limits_{x\to 2^-}[x] = 1$ whereas $\lim\limits_{x\to 2^+}[x] = 2$ so the limit does not exist

Exercise 4B

1 $f(3) = \lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^+} f(x) = 7$ so the function is continuous □

2 $f(2) = \lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x) = 3$ so the function is continuous □

3 $\lim\limits_{x\to 2^+} f(x) = \frac{x-2}{|x-2|} = 1$ and $\lim\limits_{x\to 2^-} f(x) = \frac{x-2}{|x-2|} = -1$ hence the function is not continuous.

4 $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^-} f(x) = 2a$ but $f(a) = 2a$ so the function is continuous

5 $\lim\limits_{x\to 1^-} f(x) = \frac{2}{3}$ and $\lim\limits_{x\to 1^+} f(x) = f(1) = \frac{2}{3}$, and $f(1) = \frac{2}{3}$, hence f is continuous at x=1.

$\lim_{x\to-2^-} f(x) = \infty$ and $\lim_{x\to2^+} f(x) = -\infty$ but $f(-2) = 4$
so f is not continuous at $x = -2$.

6 $\lim_{x\to-2^-} f(x) = \lim_{x\to2^+} f(x)$

$\Rightarrow (-2)^2 - 2 = 3k(-2)$

$\Rightarrow 2 = -6k \Rightarrow k = -\dfrac{1}{3}$

7 The function is already continuous for $x > 3$ and for $x < 3$

Since the functions $f_1(x) = kx^2 - k$ and $f_2(x) = 4$ are both continuous on their domains, it remains to find the value of k that insures that the function is continuous at x=3
i.e. $\lim_{x\to3^+} f(x) = \lim_{x\to3^-} f(x)$

$\Rightarrow k(3)^2 - k = 4 \Rightarrow k = \dfrac{1}{2}$

8 **a** $f(x) = \dfrac{x^2+9}{(x+3)(x-3)}$

$\therefore$ Discontinuous at $x = \pm3$

b $f(x) = \dfrac{x+1}{1-x^2} = \dfrac{1}{1-x}$ so discontinuous at $x = 1$

c Continuous

d $f(x) = \dfrac{(x+2)^2+1}{(x+5)(x-1)}$

Discontinuous at $x = 1$ and $x = -5$

e $f(x) = \dfrac{x^2}{(x-1)(x^2+x+1)}$

Discontinuous at $x = 1$

f Continuous

Exercise 4C

1 **a** $\lim_{x\to3} \dfrac{x+2}{x-2} = \dfrac{3+2}{3-2} = 5$

b $\lim_{x\to1} \dfrac{x^2+x-2}{x-1} = \lim_{x\to1}(x+2) = 3$

c $\lim_{x\to\sqrt[3]{3}} \dfrac{x^6-9}{x^3-3} = \lim_{x\to\sqrt[3]{3}} \dfrac{(x^3-3)(x^3+3)}{(x^3-3)} = 6$

d $\lim_{x\to2} \dfrac{x^2-4}{x^2-2x} = \lim_{x\to2} \dfrac{x+2}{x} = 2$

e $\lim\limits_{x\to 0}\dfrac{(x+2)(x-2)}{x(x-2)}=\lim\limits_{x\to 0}\dfrac{x+2}{x}$ limit does not exist

f $\lim\limits_{x\to 2}\dfrac{2}{2+\dfrac{2}{2-x}}=\lim\limits_{x\to 2}\dfrac{2(2-x)}{2(2-x)+2}=\lim\limits_{x\to 2}\dfrac{2(2-x)}{2(3-x)}=0$

g $\lim\limits_{x\to 0}\dfrac{(2+3x)^2-4(1+x)^2}{6x}=\lim\limits_{x\to 0}\dfrac{4+12x+9x^2-4-8x-4x^2}{6x}$

$=\lim\limits_{x\to 0}\dfrac{4x+5x^2}{6x}=\lim\limits_{x\to 0}\dfrac{4+5x}{6}=\dfrac{2}{3}$

h $\lim\limits_{x\to a}\dfrac{a^2x^2-b^2}{ax-b}=\lim\limits_{x\to a}\dfrac{(ax+b)(ax-b)}{ax-b}=\lim\limits_{x\to a}(ax+b)=a^2+b$

2 **a** $\lim\limits_{x\to\infty}f(x)=\lim\limits_{x\to\infty}\dfrac{3x}{x+3}=\lim\limits_{x\to\infty}\dfrac{3}{1+\dfrac{3}{x}}=3$

b $\lim\limits_{x\to\infty}f(x)=\lim\limits_{x\to\infty}\dfrac{-2}{1-\dfrac{1}{x^2}}=-2$

c $\lim\limits_{x\to\infty}f(x)=\lim\limits_{x\to\infty}\dfrac{-1+\dfrac{1}{x}+\dfrac{2}{x^2}}{3+\dfrac{2}{x}-\dfrac{1}{x^2}}=-\dfrac{1}{3}$

d Limit does not exist

e $\lim\limits_{x\to\infty}f(x)=\lim\limits_{x\to\infty}\dfrac{\dfrac{2}{x}+\dfrac{3}{x^2}}{5-\dfrac{2}{x^2}}=0$

f $\lim\limits_{x\to\infty}f(x)=\dfrac{\dfrac{1}{x}-\dfrac{1}{x^2}}{1+\dfrac{3}{x}-\dfrac{2}{x^2}}=0$

g Limit does not exist

h $\lim\limits_{x\to\infty}f(x)=\lim\limits_{x\to\infty}\left(\sqrt{4-\dfrac{1}{x}}+2\sqrt{1+\dfrac{3}{x}}\right)=4$

3 **a** Vertical Asymptote: $6x-1=0\Rightarrow x=\dfrac{1}{6}$

Horizontal Asymptote:

$\lim\limits_{x\to\infty}\dfrac{3x}{6x-1}=\lim\limits_{x\to\infty}\dfrac{3}{6-\dfrac{1}{x}}=\dfrac{1}{2}$ so $y=\dfrac{1}{2}$

b Vertical Asymptote: $x^2-3=0 \Rightarrow x=\pm\sqrt{3}$

Horizontal Asymptote:

$$\lim_{x\to\infty}\frac{x^2+3}{3-x^2}=\lim_{x\to\infty}\frac{1+\frac{3}{x^2}}{\frac{3}{x^2}-1}=-1 \quad \text{so} \quad y=-1$$

c Vertical Asymptote: $x^3-1=0 \Rightarrow (x-1)(x^2+x+1)=0 \Rightarrow x=1$

Horizontal Asymptote:

$$\lim_{x\to\infty}\frac{1-x-x^3}{x^3-1}=\lim_{x\to\infty}\frac{\frac{1}{x^3}-\frac{1}{x^2}-1}{1-\frac{1}{x^3}}=-1 \quad \text{so} \quad y=-1$$

d Vertical Asymptote: $x^2-2=0 \Rightarrow x=\pm\sqrt{2}$

Horizontal Asymptote:

$$\lim_{x\to\infty}\left[-\frac{5x}{x^2-2}\right]=\lim_{x\to\infty}\left[-\frac{\frac{5}{x}}{1-\frac{2}{x^2}}\right]=0 \quad \text{so} \quad y=0$$

e Vertical Asymptote: x=0

Horizontal Asymptote: None

f $r(x)=\dfrac{x^2}{2x^2-3x+1}=\dfrac{x^2}{(2x-1)(x-1)}$

Vertical Asymptotes: $x=\dfrac{1}{2}$ and $x=1$

Horizontal Asymptotes

$$\lim_{x\to\infty}\frac{x^2}{2x^2-3x+1}=\lim_{x\to\infty}\frac{1}{2-\frac{3}{x}+\frac{1}{x^2}}=\frac{1}{2}$$

Exercise 4D

1 a Divergent

b Divergent

c Convergent: $u_n=\dfrac{1}{n} \Rightarrow \lim\limits_{n\to\infty}u_n=\lim\limits_{n\to\infty}\dfrac{1}{n}=0$

d Convergent: $u_n=\dfrac{1}{3^n} \Rightarrow \lim\limits_{n\to\infty}u_n=\lim\limits_{n\to\infty}\dfrac{1}{3^n}=0$

2 a Converges: $\lim\limits_{n\to\infty}\dfrac{n+2}{n}=\lim\limits_{n\to\infty}\dfrac{1+\frac{2}{n}}{1}=1$

b Converges: $\lim\limits_{x\to\infty}\dfrac{n+2}{2n+3}=\lim\limits_{n\to\infty}\dfrac{1+\dfrac{2}{n}}{2+\dfrac{3}{n}}=\dfrac{1}{2}$

c Converges: $\lim\limits_{n\to\infty}\dfrac{n^2-n}{2n^2+\sqrt{n}}=\lim\limits_{n\to\infty}\dfrac{1-\dfrac{1}{n}}{2+\dfrac{1}{n\sqrt{n}}}=\dfrac{1}{2}$

d Diverges

e $\lim\limits_{n\to\infty}\dfrac{2n^2+1}{1-2n^3}=\lim\limits_{n\to\infty}\dfrac{\dfrac{2}{n}+\dfrac{1}{n^3}}{\dfrac{1}{n^3}-2}=0$

f $\lim\limits_{n\to\infty}\dfrac{1+n^2}{1-n^3}=\lim\limits_{n\to\infty}\dfrac{\dfrac{1}{n^3}+\dfrac{1}{n}}{\dfrac{1}{n^3}-1}=0$

3 **a** Converges since ratio of the geometric series is $\left|-\dfrac{1}{3}\right|<1$

$$S_\infty=1\cdot\frac{1}{1-\left(-\dfrac{1}{3}\right)}=\frac{3}{4}$$

b Converges since the ratio of the geometric series is $\left|\dfrac{1}{2}\right|<1$

$$S_\infty=\frac{3}{2}\cdot\frac{1}{1-\dfrac{1}{2}}=3$$

c Converges since the ratio of the geometric series is $\left|\dfrac{1}{10}\right|<1$

$$S_\infty=2\cdot\frac{1}{1-\dfrac{1}{10}}=\frac{20}{9}$$

d Converges since the ratio of both geometric series is $\left|\dfrac{3}{5}\right|<1$ and $\left|\dfrac{2}{5}\right|<1$

$$S_\infty=\frac{3}{5}\cdot\frac{1}{1-\dfrac{3}{5}}-\frac{2}{5}\cdot\frac{1}{1-\dfrac{2}{5}}=\frac{3}{2}-\frac{2}{3}=\frac{5}{6}$$

e Converges since $0<e<\pi\Rightarrow$ the ratio of the geometric series is $\left|\dfrac{e}{\pi}\right|<1$

$$S_\infty=\frac{1}{1-\dfrac{e}{\pi}}=\frac{\pi}{\pi-e}$$

f Diverges since $\pi > 3.14 > 0 \Rightarrow$ the ratio of the geometric series $\left|\frac{\pi}{3.14}\right| > 1$

4 a $2^x > 0$ for all real x so need to solve $2^x < 1$

$\Rightarrow x < 0$

b $S_\infty = \frac{42}{1-2^x} = 48$

$\Rightarrow \frac{7}{8} = 1 - 2^x$

$\Rightarrow 2^x = \frac{1}{8}$

$\Rightarrow x = -3$

5 $\left|\frac{3x}{x+1}\right| < 1 \Rightarrow |3x| < |x+1|$

Solve $|3x| = |x+1|$

$\therefore 3x = x+1$ or $-3x = x+1$

$\Rightarrow x = \frac{1}{2}$ or $x = -\frac{1}{4}$

By sketching graphs, deduce that

$-\frac{1}{4} < x < \frac{1}{2}$

6 Let $S_N = \sum_{n=0}^{N} \frac{1+2^n}{3^{n-1}}$

Then,

$$S_N = 3\sum_{n=0}^{N}\left(\frac{1}{3}\right)^n + 3\sum_{n=0}^{N}\left(\frac{2}{3}\right)^n$$

$$= 3\frac{1-\left(\frac{1}{3}\right)^{N+1}}{1-\frac{1}{3}} + 3\frac{1-\left(\frac{2}{3}\right)^{N+1}}{1-\frac{2}{3}}$$

$$= \frac{9}{2}\left[1-\left(\frac{1}{3}\right)^{N+1}\right] + 9\left[1-\left(\frac{2}{3}\right)^{N+1}\right]$$

$$\therefore S_\infty = \lim_{N\to\infty} S_N = \lim_{N\to\infty}\left\{\frac{9}{2}\left[1-\left(\frac{1}{3}\right)^{N+1}\right] + 9\left[1-\left(\frac{2}{3}\right)^{N+1}\right]\right\}$$

$$= \frac{9}{2} + 9 = \frac{27}{2}$$

so the infinite sum converges, and is equal to $\frac{27}{2}$

Exercise 4E

1 a $f'(-1) = \lim_{h\to 0}\frac{\left[2(-1+h)^2+1\right]-\left[2(-1)^2+1\right]}{h}$

$$= \lim_{h\to 0} \frac{2\left(h^2 - 2h + 1\right) + 1 - 3}{h}$$

$$= \lim_{h\to 0} \frac{2h^2 - 4h}{h} = \lim_{h\to 0}(2h - 4) = -4$$

b $f'(1) = \lim_{h\to 0} \frac{\left[1 - 3(1+h)^2\right] - \left[1 - 3(1)^2\right]}{h}$

$$= \lim_{h\to 0} \frac{1 - 3\left(h^2 + 2h + 1\right) + 2}{h}$$

$$= \lim_{h\to 0} \frac{-3h^2 - 6h}{h} = \lim_{h\to 0}(-3h - 6) = -6$$

c $f'(-1) = \lim_{h\to 0} \frac{\frac{2}{-1+h} - \frac{2}{-1}}{h}$

$$= \lim_{h\to 0} \frac{\frac{2}{h-1} + 2}{h} = \lim_{h\to 0} \frac{2 + 2(h-1)}{h(h-1)} = \lim_{h\to 0} \frac{2h}{h^2 - h}$$

$$= \lim_{h\to 0} \frac{2}{h-1} = -2$$

d $f'(-1) = \lim_{h\to 0} \frac{-(-1+h)^2 - \left[-(-1)^2\right]}{h}$

$$= \lim_{h\to 0} \frac{-h^2 + 2h - 1 + 1}{h}$$

$$= \lim_{h\to 0}(-h + 2) = 2$$

e $f'(-1) = \lim_{h\to 0} \frac{(-1+h)^3 - (-1)^3}{h}$

$$= \lim_{h\to 0} \frac{h^3 - 3h^2 + 3h - 1 + 1}{h}$$

$$= \lim_{h\to 0}\left(h^2 - 3h + 3\right) = 3$$

f $f'(0) = \lim_{h\to 0} \frac{h^2 + h - 1 - (-1)}{h}$

$$= \lim_{h\to 0}(h + 1) = 1$$

g $f'(2) = \lim_{h\to 0} \frac{\frac{1}{(2+h)^2} - \frac{1}{2^2}}{h}$

$$= \lim_{h\to 0} \frac{4 - (2+h)^2}{4h(2+h)^2} = \lim_{h\to 0} \frac{-h^2 - 4h}{4h(2+h)^2}$$

$$= \lim_{h\to 0} \frac{-h(h+4)}{4h(2+h)^2} = -\frac{1}{4} \lim_{h\to 0} \frac{h+4}{(2+h)^2}$$

$$= -\frac{1}{4}$$

h $f'(0)=\lim_{h\to 0}\dfrac{\dfrac{h}{h+1}-0}{h}=\lim_{h\to 0}\dfrac{1}{h+1}=1$

2 Gradient of line AB:

$$\frac{(1+h)^2+3-(1+3)}{h}=\frac{h^2+2h+1+3-4}{h}=h+2$$

This becomes the gradient of the tangent to $f(x)$ i.e. $f'(x)$ in the limit $h\to 0$

$\therefore f'(1)=\lim_{h\to 0}(h+2)=2$

3 **a** $f'(x)=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{\left[3(x+h)^2+2(x+h)-1\right]-\left[3x^2+2x-1\right]}{h}$$

$$=\lim_{h\to 0}\frac{3x^2+6xh+3h^2+2x+2h-1-3x^2-2x+1}{h}$$

$$=\lim_{h\to 0}\frac{6xh+3h^2+2h}{h}$$

$$=\lim_{h\to 0}(6x+2+3h)=6x+2$$

b $f'(x)=-4\Rightarrow 6x+2=-4\Rightarrow x=-1$

$\therefore\left(-1,3(-1)^2+2(-1)-1\right)=(-1,0)$

4 $f'(x)=\lim_{h\to 0}\dfrac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h}$

$$=\lim_{h\to 0}\frac{x^2-(x+h)^2}{hx^2(x+h)^2}$$

$$=\lim_{h\to 0}\frac{-2xh-h^2}{hx^2(x+h)^2}$$

$$=\lim_{h\to 0}\frac{-2x-h}{x^4+2x^3h+h^2x^2}$$

$$=\frac{-2x}{x^4}=-\frac{2}{x^3}$$

$\therefore$ If the gradient is $-\dfrac{1}{4}$,

$-\dfrac{2}{x^3}=-\dfrac{1}{4}\Rightarrow x^3=8\Rightarrow x=2$

$\therefore\left(2,\dfrac{1}{4}\right)$

Exercise 4F

1 a $f'(x)=\lim_{h\to 0}\frac{(x+h)^2+(x+h)-2-x^2-x+2}{h}$

$=\lim_{h\to 0}\frac{x^2+2xh+h^2+x+h-2-x^2-x+2}{h}$

$=\lim_{h\to 0}\frac{2xh+h^2+h}{h}$

$=\lim_{h\to 0}(2x+h+1)=2x+1$

$\therefore f'(0)=1$

b $f'(x)=\lim_{h\to 0}\frac{2-(x+h)+3(x+h)^2-2+x-3x^2}{h}$

$=\lim_{h\to 0}\frac{2-x-h+3x^2+6xh+3h^2-2+x-3x^2}{h}$

$=\lim_{h\to 0}\frac{(-1+6x)h+3h^2}{h}$

$=\lim_{h\to 0}(-1+6x+3h)=-1+6x$

$\therefore f'(-1)=-7$

c $f'(x)=\lim_{h\to 0}\frac{-\frac{2}{x+h}+\frac{2}{x}}{h}$

$=\lim_{h\to 0}\frac{-2x+2(x+h)}{hx(x+h)}=\lim_{h\to 0}\frac{-2x+2x+2h}{hx^2+h^2x}$

$=\lim_{h\to 0}\frac{2}{x^2+hx}=\frac{2}{x^2}$

$\therefore f'(1)=2$

d $f'(x)=\lim_{h\to 0}\frac{\sqrt{x+1+h}-\sqrt{x+1}}{h}$

$=\lim_{h\to 0}\frac{\sqrt{x+1+h}-\sqrt{x+1}}{h}\cdot\frac{\sqrt{x+1+h}+\sqrt{x+1}}{\sqrt{x+1+h}+\sqrt{x+1}}$

$=\lim_{h\to 0}\frac{x+1+h-(x+1)}{h\left(\sqrt{x+1+h}+\sqrt{x+1}\right)}=\lim_{h\to 0}\frac{h}{h\left(\sqrt{x+1+h}+\sqrt{x+1}\right)}$

$=\lim_{h\to 0}\frac{1}{\left(\sqrt{x+1+h}+\sqrt{x+1}\right)}=\frac{1}{2\sqrt{x+1}}$

$\therefore f'(3)=\frac{1}{4}$

e $f'(x)=\lim_{h\to 0}\frac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h}$

$$= \lim_{h\to 0}\frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} = \lim_{h\to 0}\frac{\left(\sqrt{x}-\sqrt{x+h}\right)\left(\sqrt{x}+\sqrt{x+h}\right)}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}$$

$$= \lim_{h\to 0}\frac{x-(x+h)}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}$$

$$= \lim_{h\to 0}\frac{-1}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}$$

$$= \frac{-1}{\sqrt{x}\sqrt{x}\left(\sqrt{x}+\sqrt{x}\right)} = -\frac{1}{2x\sqrt{x}}$$

$$\therefore f'(9) = -\frac{1}{2\cdot 9\cdot 3} = -\frac{1}{54}$$

f $f'(x) = \lim_{h\to 0}\frac{(x+h)^3-1-x^3+1}{h}$

$$= \lim_{h\to 0}\frac{3hx^2+3h^2x+h^3}{h}$$

$$= \lim_{h\to 0}\left(3x^2+3hx+h^2\right)$$

$$= 3x^2$$

$$\therefore f'(1) = 3$$

2 a $v_{avg} = \frac{8+2(a+h)^2-8-2a^2}{h} = \frac{4ah+2h^2}{h} = 4a+2h$

b In the limit $h\to 0$, $v_{avg}\to v_A = \lim_{h\to 0}(4a+2h) = 4a$

3 a $v = \lim_{h\to 0}\frac{10(t+h)^2-(t+h)^3-10t^2+t^3}{h}$

$$= \lim_{h\to 0}\frac{10t^2+20th+10h^2-t^3-3ht^2-3h^2t-h^3-10t^2+t^3}{h}$$

$$= \lim_{h\to 0}\frac{20th+10h^2-3ht^2-3h^2t-h^3}{h}$$

$$= \lim_{h\to 0}\left(20t-3t^2+10h-h^2-3ht\right)$$

$$= 20t-3t^2$$

b $v(1) = 17$, $\mathrm{v}(10) = -100$

The sign indicates the direction the particle moves in. At $t=1$, the particle is moving in the positive direction and in the opposite direction at $t=10$

Exercise 4G

1 a $f'(1) = \lim_{h\to 0}\frac{2(1+h)^2-(1+h)+1-2+1-1}{h}$

$$= \lim_{h\to 0}\frac{2\left(h^2+2h+1\right)-(h+1)-1}{h} = \lim_{h\to 0}\frac{2h^2+3h}{h}$$

$$= \lim_{h\to 0}(2h+3) = 3$$

b $y = f(1) = 2$

$y - 2 = 3(x-1) \Rightarrow y = 3x - 1$

c The normal has gradient $-\frac{1}{3}$ and also passes through $(1,2)$

$\therefore y - 2 = -\frac{1}{3}(x-1) \Rightarrow y = -\frac{1}{3}x + \frac{7}{3}$

2 $f'(x) = \lim_{h\to 0} \dfrac{\frac{1}{2-(x+h)} - \frac{1}{2-x}}{h} = \lim_{h\to 0} \dfrac{2-x-(2-x-h)}{h(2-x)(2-x-h)}$

$= \lim_{h\to 0} \dfrac{h}{h(2-x)(2-x-h)}$

$= \lim_{h\to 0} \dfrac{1}{(2-x)(2-x-h)}$

$= \dfrac{1}{(2-x)^2}$

$\therefore f'(x) = 1 \Rightarrow \dfrac{1}{(2-x)^2} = 1 \Rightarrow (2-x)^2 = 1$

$\Rightarrow x = 1$ or $x = 3$

At $x = 1$, $y = \dfrac{1}{2-1} = 1$ so the tangent here is

$y - 1 = x - 1 \Rightarrow y = x$

At $x = 3$, $y = \dfrac{1}{2-3} = -1$ so the tangent here is

$y - (-1) = x - 3 \Rightarrow y = x - 4$

3 **a** $\dfrac{dy}{dx} = \lim_{h\to 0} \dfrac{2(x+h)^2 - 1 - 2x^2 + 1}{h} = \lim_{h\to 0} \dfrac{4xh + 2h^2}{h}$

$= \lim_{h\to 0}(4x + 2h) = 4x$

so there exists a horzontal tangent at $x = 0 \Rightarrow (0,-1)$

b $\dfrac{dy}{dx} = \lim_{h\to 0} \dfrac{2 - 3(x+h) - (x+h)^2 - 2 + 3x + x^2}{h}$

$= \lim_{h\to 0} \dfrac{2 - 3x - 3h - x^2 - 2xh - h^2 - 2 + 3x + x^2}{h}$

$= \lim_{h\to 0} \dfrac{(-3-2x)h - h^2}{h}$

$= \lim_{h\to 0}[-3 - 2x - h] = -3 - 2x$

so there is a horizontal tangent at $x = -\dfrac{3}{2}$

i.e. at the point $\left(-\dfrac{3}{2}, \dfrac{17}{4}\right)$

c $\dfrac{dy}{dx} = \lim_{h\to 0} \dfrac{(x+h)^3 - 1 - x^3 + 1}{h}$

$$= \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-1-x^3+1}{h}$$

$$= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}$$

$$= \lim_{h\to 0}\left(3x^2+3xh+h^2\right)=3x^2$$

so there is a horizontal tangent when $x=0$

i.e. at the point $(0,-1)$

d $$\frac{dy}{dx}=\lim_{h\to 0}\frac{(x+h)^3-3(x+h)-x^3+3x}{h}$$

$$= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3-3h}{h}$$

$$= \lim_{h\to 0}\left(3x^2+3xh+h^2-3\right)$$

$$=3x^2-3$$

so there is a horizontal tangent at $x=\pm 1$

i.e. at the points $(\pm 1,\mp 2)$

4 $$f'(1)=\lim_{h\to 0}\frac{1+h+\frac{1}{1+h}-1-1}{h}=\lim_{h\to 0}\frac{h-1+\frac{1}{1+h}}{h}$$

$$= \lim_{h\to 0}\frac{h^2-1+1}{h}$$

$$= \lim_{h\to 0}h=0$$

At $x=1,\ y=1+\frac{1}{1}=2$

$\therefore$ Tangent: $y=2$

Normal: $x=1$

Exercise 4H

1 **a** $y=(3x-1)^2=9x^2-6x+1$

$$\frac{dy}{dx}=2\cdot 9x-6=18x-6$$

b $$\frac{dy}{dx}=5\cdot 3x^4-2\cdot 4x+2=15x^4-8x+2$$

c $$\frac{dy}{dx}=2\cdot\frac{1}{4}x+\frac{2}{3}=\frac{1}{2}x+\frac{2}{3}$$

d $y=5x^5-4x^3+x-\frac{1}{4}x^{-3}-\frac{1}{5}x^{-5}$

$$\frac{dy}{dx}=25x^4-12x^2+1+\frac{3}{4}x^{-4}+x^{-6}$$

$$=25x^4-12x^2+1+\frac{3}{4x^4}+\frac{1}{x^6}$$

e $y=\frac{3-2x^3+x^4}{x}=3x^{-1}-2x^2+x^3$

$$\frac{dy}{dx} = -3x^{-2} - 4x + 3x^2$$
$$= -\frac{3}{x^2} - 4x + 3x^2$$

f $y = \sqrt{x} = x^{\frac{1}{2}}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

g $y = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2x\sqrt{x}}$$

h $y = \sqrt[5]{x^2} = x^{\frac{2}{5}}$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{5}x^{-\frac{3}{5}} = \frac{2}{5\sqrt[5]{x^3}}$$

i $y = \frac{2}{\sqrt[3]{x}} - \frac{3}{\sqrt{x^5}} = 2x^{-\frac{1}{3}} - 3x^{-\frac{5}{2}}$

$$\Rightarrow \frac{dy}{dx} = -\frac{2}{3}x^{-\frac{4}{3}} + \frac{15}{2}x^{-\frac{7}{2}} = -\frac{2}{3\sqrt[3]{x^4}} + \frac{15}{2\sqrt{x^7}}$$

j $y = \left(1+\sqrt{x}\right)\left(3-\sqrt[3]{x}\right) = \left(1+x^{\frac{1}{2}}\right)\left(3-x^{\frac{1}{3}}\right)$

$$= 3 - x^{\frac{1}{3}} + 3x^{\frac{1}{2}} - x^{\frac{5}{6}}$$
$$\Rightarrow \frac{dy}{dx} = -\frac{1}{3}x^{-\frac{2}{3}} + \frac{3}{2}x^{-\frac{1}{2}} - \frac{5}{6}x^{-\frac{1}{6}}$$

2 $y = -2\left(x^2 + 3x\right) = -2x^2 - 6x$

$\frac{dy}{dx} = -4x - 6$

$\therefore$ At $x = 1$,

$y = -8$ and $\frac{dy}{dx} = -10$

So the equation of the tangent at $(1, -8)$ is

$y - (-8) = -10(x - 1)$

$\Rightarrow y = -10x + 2$

3 $y = \frac{x-3}{x} = 1 - 3x^{-1}$

$\therefore \frac{dy}{dx} = 3x^{-2}$

Therefore at $x = -1$,

$y = 4$ and $\frac{dy}{dx} = 3$

$\Rightarrow$ The gradient of the normal is $-\frac{1}{3}$

So the equation of the normal at $(-1,4)$ is

$y-4=-\frac{1}{3}(x-(-1))$

$\Rightarrow y=-\frac{1}{3}x+\frac{11}{3}$

$\Rightarrow x+3y-11=0$

4 $f'(x)=15x^2+24x-7$

Therefore at $x=1$,

$y=f(1)=10$ and $\frac{dy}{dx}=f'(1)=32$

So the tangent at $(1,10)$ is

$y-10=32(x-1)$

$\Rightarrow y=32x-22$

At $x=-1$,

$y=f(-1)=14$ and $\frac{dy}{dx}=f'(-1)=-16$

$y-14=-16(x-(-1))$

$\Rightarrow y=-16x-2$

5 $f'(x)=3x^2-10x+5$

$\therefore f'(x)=2$

$\Rightarrow 3x^2-10x+5=2$

$\Rightarrow 3x^2-10x+3=0$

$\Rightarrow (3x-1)(x-3)=0$

so $x=\frac{1}{3}$ or $x=3$

If $x=\frac{1}{3}$, $y=f\left(\frac{1}{3}\right)=-\frac{77}{27}$

so $y+\frac{77}{27}=2\left(x-\frac{1}{3}\right)$

$\Rightarrow y=2x-\frac{77}{27}-\frac{18}{27}=2x-\frac{95}{27}$

If $x=3$, $y=-7$

$\therefore y-(-7)=2(x-3)$

$\Rightarrow y=2x-13$

6 $f'(x)=2x-3$

$\therefore$ The normal at $x=1$ has gradient $-\frac{1}{2-3}=1$

$y=f(1)=-1$

$\therefore y-(-1)=1(x-1)\Rightarrow y=x-2$

$\Rightarrow x^2-3x+1=x-2$

$\Rightarrow x^2-4x+3=0$

$\Rightarrow (x-3)(x-1)=0$

$\Rightarrow x=1$ or $x=3$

$\therefore$ The other point is $(3,f(3))=(3,1)$

7 $f'(x) = 3x^2 + 2x + 1$

The line has gradient $-\frac{1}{2}$

So, set $f'(x) = 2$

$\therefore 3x^2 + 2x + 1 = 2$

$\Rightarrow 3x^2 + 2x - 1 = 0$

$\Rightarrow (3x-1)(x+1) = 0$

$\Rightarrow \ x = -1$ or $x = \frac{1}{3}$

If $x = -1$,

$y = f(-1) = -2$

$\therefore y - (-2) = 2(x - (-1))$

$\Rightarrow y = 2x$

If $x = \frac{1}{3}$,

$y = f\left(\frac{1}{3}\right) = \frac{1}{27} + \frac{1}{9} + \frac{1}{3} - 1 = \frac{1+3+9-27}{27} = -\frac{14}{27}$

$\therefore y - \left(-\frac{14}{27}\right) = 2\left(x - \frac{1}{3}\right)$

$\Rightarrow 27y + 14 = 54x - 18$

$\Rightarrow -54x + 27y + 32 = 0$

Exercise 4I

1 **a** Let $u = 4x - 3$, then $y = u^5$

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (5u^4)(4) = 20u^4 = 20(4x-3)^4$$

b Let $u = 1 - 4x$, then $y = u^{\frac{1}{2}}$

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left(\frac{1}{2}u^{-\frac{1}{2}}\right)(-4) = -2u^{-\frac{1}{2}} = -2(1-4x)^{-\frac{1}{2}}$$

c $y = \frac{2 - x^2 - 3x^5}{x} = 2x^{-1} - x - 3x^4$

$$\therefore \frac{dy}{dx} = -2x^{-2} - 1 - 12x^3 = -\frac{2 + x^2 + 12x^5}{x^2}$$

d Let $u = 1 - 3x^2$, then $y = -2u^{-\frac{1}{2}}$

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = u^{-\frac{3}{2}}(-6x) = -\frac{6x}{(1-3x^2)^{\frac{3}{2}}}$$

e $y = \left(\frac{1-\sqrt{x}}{x}\right)^3 = \left(x^{-1} - x^{-\frac{1}{2}}\right)^3$

Let $u = x^{-1} - x^{-\frac{1}{2}}$, then $y = u^3$

$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 3u^2\left(-x^{-2} + \frac{1}{2}x^{-\frac{3}{2}}\right) = 3\left(x^{-1} - x^{-\frac{1}{2}}\right)^2\left(-x^{-2} + \frac{1}{2}x^{-\frac{3}{2}}\right)$

$= 3\left(\frac{1}{x} - \frac{1}{\sqrt{x}}\right)^2\left(-\frac{1}{x^2} + \frac{1}{2x\sqrt{x}}\right)$

$= 3\left(\frac{1-\sqrt{x}}{x}\right)^2\left(\frac{-2+\sqrt{x}}{2x^2}\right)$

$= \frac{3\left(1-\sqrt{x}\right)^2\left(\sqrt{x}-2\right)}{2x^4}$

f Let $u = 2x^2 - 4$, then $y = u^{\frac{1}{3}}$

$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left(\frac{1}{3}u^{-\frac{2}{3}}\right)(4x) = \frac{4x}{3\left(2x^2-4\right)^{\frac{2}{3}}} = \frac{2^{\frac{4}{3}}x}{3\left(x^2-2\right)^{\frac{2}{3}}}$

2 $\frac{dy}{dx} = 6x - 12x^2$

Therefore the gradient at $x = 1$ is -6

$y = f(1) = -1$

$\therefore y - (-1) = -6(x-1)$

$\Rightarrow y = -6x + 5$

3 $y = 1 - \frac{2}{x} = 1 - 2x^{-1}$

$\therefore \frac{dy}{dx} = 2x^{-2}$

The gradient at $x = -1 \Rightarrow$ the gradient of the normal

to the curve at this point is $-\frac{1}{2}$

$y = f(-1) = \frac{-1-2}{-1} = 3$

$\therefore y - 3 = -\frac{1}{2}\left(x - (-1)\right)$

$\Rightarrow y = -\frac{1}{2}x + \frac{5}{2}$

$\Rightarrow x + 2y - 5 = 0$

4 $y = \sqrt{2-\sqrt{x}}$

Let $u = 2 - \sqrt{x}$, then $y = u^{\frac{1}{2}}$

$\Rightarrow \frac{du}{dx} = -\frac{1}{2\sqrt{x}}$

$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left(\frac{1}{2}u^{-\frac{1}{2}}\right)\left(-\frac{1}{2\sqrt{x}}\right) = -\frac{1}{4\sqrt{x}\sqrt{2-\sqrt{x}}}$

5 $f'(x) = \frac{20}{x^2}$ and $g'(x) = 5$

$\therefore f'(x) = g'(x) \Rightarrow \frac{20}{x^2} = 5$

$\Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

6 **a** $f'(x) = 15ax^2 - 4bx + 4c$

b $f'(x) \geq 0 \Rightarrow \Delta \leq 0$

$\therefore (-4b)^2 - 4(15a)(4c) \leq 0$

$\Rightarrow 16b^2 \leq 240ac$

$\Rightarrow b^2 \leq 15ac$

7 **i** Let $f(x)$ be an even function i.e. $f(-x) = f(x)$

Let $u = -x$, then $f(-x) = f(u)$

$\therefore \frac{d}{dx}[f(x)] = f'(x)$

and since $f(x)$ is an even function,

$\frac{d}{dx}[f(x)] = \frac{d}{dx}[f(-x)] = \frac{d}{dx}[f(u)] = \frac{du}{dx}\frac{d}{du}[f(u)] = -f'(u) = -f'(-x)$

$\therefore f'(x) = -f'(-x) \Rightarrow f'(-x) = -f(x)$

i.e. the derivative of $f(x)$ is an odd function

ii Let $g(x)$ be an odd function i.e. $g(-x) = -g(x)$

Let $u = -x$, then $g(-x) = g(u)$

Then,

$\frac{d}{dx}[g(-x)] = \frac{d}{dx}[-g(x)] = -\frac{d}{dx}[g(x)] = -g'(x)$

and also

$\frac{d}{dx}[g(-x)] = \frac{d}{dx}[g(u)] = \frac{du}{dx}\frac{d}{du}[g(u)] = -g'(u) = -g'(-x)$

$\therefore -g'(x) = -g'(-x)$

$\Rightarrow g'(-x) = g'(x)$

i.e. the derivative of $g(x)$ is an even function

Exercise 4J

1 Let $u = 2x - 3$ and $v = (x+3)^3$

Then,

$\frac{du}{dx} = 2$ and $\frac{dv}{dx} = 3(x+3)^2$

$\Rightarrow \frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx} = 2(x+3)^3 + (2x-3)\cdot 3(x+3)^2$

$= (x+3)^2[2(x+3) + 3(2x-3)]$

$= (x+3)^2(8x-3)$

2 Let $u = (2x+3)^2$ and $v = (3-x)^3$

Then,

$\frac{du}{dx} = 4(2x+3)$ and $\frac{dv}{dx} = -3(3-x)^2$

$\Rightarrow \frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx} = 4(2x+3)(3-x)^3 + (2x+3)^2 \cdot \left[-3(3-x)^2\right]$

$= (2x+3)(3-x)^2\left[4(3-x) - 3(2x+3)\right]$

$= (2x+3)(3-x)^2(3-10x)$

3 $y = \frac{x-1}{x+1} = (x-1)(x+1)^{-1}$

Let $u = x-1$ and $v = (x+1)^{-1}$

Then,

$\frac{du}{dx} = 1$ and $\frac{dv}{dx} = -(x+1)^{-2}$

$\Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$= (x-1)\left[-(x+1)^{-2}\right] + (x+1)^{-1}$

$= \frac{1}{x+1} - \frac{x-1}{(x+1)^2}$

$= \frac{x+1-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$

4 $y = x(2-3x)^{\frac{1}{2}}$

Let $u = x$ and $v = (2-3x)^{\frac{1}{2}}$

Then,

$\frac{du}{dx} = 1$ and $\frac{dv}{dx} = -\frac{3}{2}(2-3x)^{-\frac{1}{2}}$

$\Rightarrow \frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$

$= (2-3x)^{\frac{1}{2}} - \frac{3x}{2}(2-3x)^{-\frac{1}{2}}$

$= \frac{2(2-3x)-3x}{2\sqrt{2-3x}} = \frac{4-9x}{2\sqrt{2-3x}}$

5 $y = \frac{1}{x^3-2x^2+3x+1} = \left(x^3-2x^2+3x+1\right)^{-1}$

$\Rightarrow \frac{dy}{dx} = -\left(3x^2-4x+3\right)\left(x^3-2x^2+3x+1\right)^{-2}$

$= -\frac{3x^2-4x+3}{\left(x^3-2x^2+3x+1\right)^2}$

6 $y = (x+1)^4(2-3x)^{\frac{2}{3}}$

Let $u = (x+1)^4$ and $v = (2-3x)^{\frac{2}{3}}$

Then,

$\dfrac{du}{dx} = 4(x+1)^3$ and $\dfrac{dv}{dx} = -2(2-3x)^{-\frac{1}{3}}$

$\therefore \dfrac{dy}{dx} = u\dfrac{dv}{dx} + \dfrac{du}{dx}v$

$= 2(x+1)^3(2-3x)^{-\frac{1}{3}}\left[2(2-3x) - (x+1)\right]$

$= 2(x+1)^3(2-3x)^{-\frac{1}{3}}(3-7x)$

$= \dfrac{2(x+1)^3(3-7x)}{\sqrt[3]{2-3x}}$

7 $y = (2x-1)^3(4-x)^{-2}$

Let $u = (2x-1)^3$ and $v = (4-x)^{-2}$

Then,

$\dfrac{du}{dx} = 6(2x-1)^2$ and $\dfrac{dv}{dx} = 2(4-x)^{-3}$

$\therefore \dfrac{dy}{dx} = \dfrac{du}{dx}v + u\dfrac{dv}{dx} = 6(2x-1)^2(4-x)^{-2} + 2(2x-1)^3(4-x)^{-3}$

$= 2(2x-1)^2(4-x)^{-3}\left[3(4-x) + (2x-1)\right]$

$= 2(2x-1)^2(4-x)^{-3}(11-x)$

8 $y = \dfrac{1-2x}{\sqrt{3x^2+2}} = (1-2x)(3x^2+2)^{-\frac{1}{2}}$

Let $u = 1-2x$ and $v = (3x^2+2)^{-\frac{1}{2}}$

Then,

$\dfrac{du}{dx} = -2$ and $\dfrac{dv}{dx} = -3x(3x^2+2)^{-\frac{3}{2}}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{du}{dx}v + u\dfrac{dv}{dx}$

$= -2(3x^2+2)^{-\frac{1}{2}} - 3x(1-2x)(3x^2+2)^{-\frac{3}{2}}$

$= -(3x^2+2)^{-\frac{3}{2}}\left[2(3x^2+2) + 3x(1-2x)\right]$

$= -(3x^2+2)^{-\frac{3}{2}}(3x+4)$

9 $y = \dfrac{x^2+1}{x^2-3} = (x^2+1)(x^2-3)^{-1}$

Let $u = x^2+1$ and $v = (x^2-3)^{-1}$

Then,

$\dfrac{du}{dx} = 2x$ and $\dfrac{dv}{dx} = -2x(x^2-3)^{-2}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{du}{dx}v + u\dfrac{dv}{dx}$

$= 2x\left(x^2-3\right)^{-1} - 2x\left(x^2+1\right)\left(x^2-3\right)^{-2}$

$= 2x\left(x^2-3\right)^{-2}\left[\left(x^2-3\right)-\left(x^2+1\right)\right]$

$= 2x\left(x^2-3\right)^{-2}(-4)$

$= -8x\left(x^2-3\right)^{-2}$

At $x = 1$,

$\frac{dy}{dx} = (-8)(-2)^{-2} = -2$

Therefore the tangent has gradient -2 and the normal has gradient $\frac{1}{2}$

Tangent: $y - 0 = -2(x-1) \Rightarrow y = -2x + 2$

Normal: $y - 0 = \frac{1}{2}(x-1) \Rightarrow y = \frac{1}{2}x - \frac{1}{2}$

10 a $y = (x+1)^{\frac{1}{2}}(3-x)^2$

Let $u = (x+1)^{\frac{1}{2}}$ and $v = (3-x)^2$

Then,

$\frac{du}{dx} = \frac{1}{2}(x+1)^{-\frac{1}{2}}$ and $\frac{dv}{dx} = -2(3-x)$

$\therefore \frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$

$= \frac{1}{2}(x+1)^{-\frac{1}{2}}(3-x)^2 - 2(x+1)^{\frac{1}{2}}(3-x)$

$= \frac{1}{2}(x+1)^{-\frac{1}{2}}(3-x)\left[(3-x)-4(x+1)\right]$

$= \frac{1}{2}(x+1)^{-\frac{1}{2}}(3-x)(-1-5x)$

$= \frac{-(3-x)(5x+1)}{2\sqrt{x+1}}$

b $\frac{dy}{dx} = 0 \Rightarrow x = 3$ or $x = -\frac{1}{5}$

Exercise 4K

1 a $\frac{dy}{dx} = \frac{(5-x)(3)-(1+3x)(-1)}{(5-x)^2} = \frac{16}{(5-x)^2}$

b $\frac{dy}{dx} = \frac{(2-x)\left(\frac{1}{2\sqrt{x}}\right) - \sqrt{x}(-1)}{(2-x)^2} = \frac{(2-x)+\sqrt{x}\left(2\sqrt{x}\right)}{2\sqrt{x}(2-x)^2} = \frac{2+x}{2\sqrt{x}(2-x)^2}$

c $\frac{dy}{dx} = \frac{\sqrt{1-x^2}(2)-(1+2x)\frac{-x}{\sqrt{1-x^2}}}{1-x^2} = \frac{2\sqrt{1-x^2}+\frac{x(1+2x)}{\sqrt{1-x^2}}}{1-x^2}$

$$= \frac{2\left(1-x^2\right)+x\left(1+2x\right)}{\left(1-x^2\right)^{\frac{3}{2}}}$$

$$= \frac{2+x}{\left(1-x^2\right)^{\frac{3}{2}}}$$

d $\dfrac{dy}{dx} = \dfrac{\left(x^2+1\right)(3)-\left(1+3x\right)\left(2x\right)}{\left(x^2+1\right)^2} = \dfrac{3-2x-3x^2}{\left(x^2+1\right)^2}$

2 **a** $u = x^2-2; \dfrac{du}{dx} = 2x; v = x^3-1; \dfrac{dv}{dx} = 3x^2$

$$\frac{dy}{dx} = \frac{\left(x^3-1\right)\left(2x\right)-\left(x^2-2\right)\left(3x^2\right)}{\left(x^3-1\right)^2} = \frac{2x^4-2x-3x^4+6x^2}{\left(x^3-1\right)^2}$$

$$= \frac{-2x+6x^2-x^4}{\left(x^3-1\right)^2}$$

b $u = x^2+2x; \dfrac{du}{dx} = 2x+2; v = \left(x^3+1\right)^{\frac{1}{2}}; \dfrac{dv}{dx} = \dfrac{3x^2}{2}\left(x^3+1\right)^{-\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{2\left(x+1\right)\left(x^3+1\right)^{\frac{1}{2}}-\dfrac{3x^2}{2}\left(x^3+1\right)^{-\frac{1}{2}}\left(x^2+2x\right)}{x^3+1}$$

$$\frac{dy}{dx} = \frac{\left(x^3+1\right)^{-\frac{1}{2}}\left[2\left(x+1\right)\left(x^3+1\right)-\dfrac{3x^2\left(x^2+2x\right)}{2}\right]}{x^3+1}$$

$$\frac{dy}{dx} = \frac{2\left(x^3+1\right)^{-\frac{1}{2}}\left[\left(x+1\right)\left(x^3+1\right)-\dfrac{3x^2\left(x^2+2x\right)}{4}\right]}{x^3+1}$$

$$\frac{dy}{dx} = \frac{x^4-2x^3+4x+4}{2\left(x^3+1\right)^{\frac{3}{2}}}$$

c $y = \dfrac{1}{\sqrt{x^4-x^5+2x^6}} = \left(x^4-x^5+2x^6\right)^{-\frac{1}{2}}$

$u = 1; \dfrac{du}{dx} = 0; v = \left(x^4-x^5+2x^6\right)^{-\frac{1}{2}}; \dfrac{dv}{dx} = -\dfrac{1}{2}\left(x^4-x^5+2x^6\right)^{-\frac{3}{2}}\left(4x^3-5x^4+12x^5\right)$

$$\therefore \frac{dy}{dx} = \frac{0-\left[-\dfrac{1}{2}\left(x^4-x^5+2x^6\right)^{-\frac{2}{3}}(4x^3-5x^4+12x^5)\right]}{\left(x^4-x^5+2x^6\right)^2}$$

$$= \frac{-\left(12x^2-5x+4\right)}{2x^3\left(2x^2-x+1\right)^{\frac{3}{2}}}$$

d $\dfrac{dy}{dx} = \dfrac{\left(1+\sqrt{x}\right)\left(-\dfrac{1}{2\sqrt{x}}\right)-\left(1-\sqrt{x}\right)\left(\dfrac{1}{2\sqrt{x}}\right)}{\left(1+\sqrt{x}\right)^2}$

$$= \frac{-\left(1+\sqrt{x}\right)-\left(1-\sqrt{x}\right)}{2\sqrt{x}\left(1+\sqrt{x}\right)^2} = \frac{-2}{2\sqrt{x}\left(1+\sqrt{x}\right)^2} = -\frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)^2}$$

e $$\frac{dy}{dx} = \frac{3\sqrt{\sqrt{x}-1}-(1+3x)\left(\dfrac{1}{4\sqrt{x}\sqrt{\sqrt{x}-1}}\right)}{\sqrt{x}-1}$$

$$= \frac{12\sqrt{x}\left(\sqrt{x}-1\right)-(1+3x)}{4\sqrt{x}\left(\sqrt{x}-1\right)^{\frac{3}{2}}}$$

$$= \frac{9x-12\sqrt{x}-1}{4\sqrt{x}\left(\sqrt{x}-1\right)^{\frac{3}{2}}}$$

f $$\frac{dy}{dx} = \frac{2}{3}\frac{1}{(x+2)^2}\left(1-\frac{1}{2+x}\right)^{-\frac{1}{3}} = \frac{2}{3(x+1)^{\frac{1}{3}}(x+2)^{\frac{5}{3}}}$$

3 $$\frac{dy}{dx} = \frac{\left(x^2+1\right)(4)-(4x)(2x)}{\left(x^2+1\right)^2} = \frac{-4x^2+4}{\left(x^2+1\right)^2}$$

At $x=0$, $y=0$ and $\frac{dy}{dx}=4$ so the normal has gradient $-\frac{1}{4}$

$\therefore$ The normal has equation $y=-\frac{1}{4}x$

4 $y = 8\left(4+x^2\right)^{-1}$

$$\Rightarrow \frac{dy}{dx} = -8(2x)\left(4+x^2\right)^{-2} = -\frac{16x}{\left(4+x^2\right)^2}$$

At $x=1$, $y=\frac{8}{5}$ and $\frac{dy}{dx}=-\frac{16}{25}$

$\therefore$ The normal has gradient $\frac{25}{16}$ and its equation is

$$y-\frac{8}{5} = \frac{25}{16}(x-1) \Rightarrow y = \frac{25}{16}x+\frac{3}{80}$$

5 $y = x^2+x+1+\frac{1}{x}$

$$\frac{dy}{dx} = 2x+1-\frac{1}{x^2} = \frac{2x^3+x^2-1}{x^2}$$

$$\frac{dy}{dx} = 2$$

$\Rightarrow 2x^3+x^2-1 = 2x^2$

$\Rightarrow 2x^3-x^2-1 = (x-1)\left(2x^2+x+1\right) = 0$

$x=1$ is the only real solution and the corresponding coordinate is $(1,4)$

Exercise 4L

1 $f(x) = 1 - 4x - x^{-1}$

$f'(x) = -4 + x^{-2}$

$f''(x) = -2x^{-3} = -\dfrac{2}{x^3}$

2 $f(x) = 3x^5 - 2x^2 + 1$

$f'(x) = 15x^4 - 4x$

$f''(x) = 60x^3 - 4$

$f'''(x) = 180x^2$

$f^{(4)}(x) = 360x$

$\therefore f^{(4)}(1) = 360$

3 $y = 1 - 3ax + 3a^2x^2 - a^3x^3$

$\dfrac{d^3y}{dx^3} = -6a^3 = 162 \Rightarrow a^3 = -27$

$\therefore a = -3$

4 $f'(x) = \dfrac{x^3}{3} - 4x + 5$

$f''(x) = x^2 - 4 = 0 \Rightarrow x = \pm 2$

5 $\dfrac{dy}{dx} = 4x^3 - 12x^2 + 16$

$\dfrac{d^2y}{dx^2} = 12x^2 - 24x$

$\dfrac{d^3y}{dx^3} = 24x - 24$

$\dfrac{d^2y}{dx^2} = \dfrac{d^3y}{dx^3} \Rightarrow 12x^2 - 24x = 24x - 24$

$\Rightarrow x^2 - 4x + 2 = 0$

$\Rightarrow (x-2)^2 = 2 \Rightarrow x = 2 \pm \sqrt{2}$

6 $f(x) = x^4 + px^2 + qx + r$

$f'(x) = 4x^3 + 2px + q$

$f''(x) = 12x^2 + 2p$

$f''(-1) = 16 \Rightarrow 12 + 2p = 16 \Rightarrow p = 2$

$f'(-1) = -16 \Rightarrow -4 - 2p + q = 16 \Rightarrow q = 16 + 4 + 2p = 24 \Rightarrow q = 24$

$f(-1) = 1 + p - q + r = 16 \Rightarrow r = 16 - 1 - p + q = 37 \Rightarrow r = 37$

7 **a** Notice that $s(t) = 4(t-2)^5$

$v(t) = s'(t)$

$v(t) = 20(t-2)^4$

$\therefore v(3) = 20 \text{ ms}^{-1}$

b $a(t) = v'(t) = s''(t)$

$a(t) = 80(t-2)^3$

$\therefore a(3) = 80 \text{ ms}^{-2}$

c $a'(t) = v''(t) = s'''(t)$

$a'(t) = 240(t-2)^2$

$a'(1) = 240 \text{ ms}^{-3}$ □

8 $f(x) = \dfrac{1}{x}$

$f'(x) = -\dfrac{1}{x^2}$

$f''(x) = \dfrac{2}{x^3}$

$f'''(x) = -\dfrac{6}{x^4}$

$f^{(4)}(x) = \dfrac{24}{x^5}$

$f^{(5)}(x) = -\dfrac{120}{x^6}$

Conjecture that $P(n): f^{(n)}(x) = (-1)^n \dfrac{n!}{x^{n+1}}$

The statement $P(1)$ is true (see above)

Assume that $P(k)$ is true for some $k \in \mathbb{Z}^+$

i.e. that $f^{(k)}(x) = (-1)^k \dfrac{k!}{x^{k+1}}$

Then,

$f^{(k+1)}(x) = \dfrac{d}{dx} f^{(k)}(x)$

$= \dfrac{d}{dx}(-1)^k \dfrac{k!}{x^{k+1}}$

$= (-1)^k k! \dfrac{d}{dx}\left[x^{-(k+1)}\right]$

$= -(-1)^k k!(k+1)x^{-k-2}$

$= (-1)^{k+1} \dfrac{(k+1)!}{x^{k+1+1}}$

so $P(k) \Rightarrow P(k+1)$

Therefore, $P(1)$ has been shown to be true and if $P(k)$ is true for some $k \in \mathbb{Z}^+$ then so is $P(k+1)$. Hence, the statement is true for all positive integers by the principle of mathematical induction

9 $f(x) = (x-1)^4\left[(x+1)^4(1-2x^3)^3\right]$

Note that all derivatives up to and including the third derivative of $(x-1)^4$ will include a factor of $(x-1)$ and therefore disappear upon being evaluated at $x=1$. Therefore, we only need to include the first term in Leibniz:

$$f'(1) = \frac{d^4\left[(x-1)^4\right]}{dx^4}\frac{d^0}{dx^0}\left[(x+1)^4\left(1-2x^3\right)^3\right]\Bigg|_{x=1}$$

$$= 24\left[(x+1)^4\left(1-2x^3\right)^3\right]\Bigg|_{x=1} = -384$$

Exercise 4M

1 a $f'(x)=0 \Rightarrow f$ has a stationary point at x. So, $x=-3,-1,2,4$

b $f'(x)>0 \Rightarrow f$ is increasing, so $x<-3, -1<x<2, x>4$

c $f'(x)<0 \Rightarrow f$ is decreasing, so $-3<x<-1, 2<x<4$

d By inspection of the graph of *f*, $x\le -3, -1\le x\le 2, 4\le x$

e By insprection of the graph of *f*, f' is decreasing at $x\le -1, 2\le x\le 3$

2 a i By inspection of the graph of *f*, $x\le -3, x\ge 4$

ii By inspection of the graph of *f*, $-3\le x\le 4$

iii By inspection of the graph of *f*, $x=-3,4$

b i By inspection of the graph of *f*, $x\le -1, 0\le x\le 1, 2\le x$

ii By inspection of the graph of *f*, $-1\le x\le 0, 1\le x\le 2$

iii By inspection of the graph of *f*, $x=-1,0,1,2$

3 a $y=2x^3-6x^2+3$

i $\frac{dy}{dx}=6x^2-12x=0 \Rightarrow x(x-2)=0 \Rightarrow x=0$ or $x=2$

$f'(-0.5)>0, f'(0.5)<0 \Rightarrow (0,3)$ is a maximum.
$f'(1.5)<0, f'(2.5)>0 \Rightarrow (2,-5)$ is a minimum.

ii $6x^2-12>0 \ \Rightarrow x<0$ or $x>2$

iii $6x^2-12<0 \Rightarrow 0<x<2$

iv Both turning points are local.

b $y=-3x^4+2x^3+3x^2-4$

i □ $\frac{dy}{dx}=-12x^3+6x^2+6x=6x\left(-2x^2+x+1\right)=6x(-2x-1)(x-1)$

$\therefore \frac{dy}{dx} = 0$ when $x = -\frac{1}{2}$, $x = 0$ or $x = 1$

$f'(-1) > 0; f'(-\frac{1}{4}) < 0 \Rightarrow \left(-\frac{1}{2}, -\frac{59}{16}\right)$ is a maximum

$f'(-0.5) < 0; f'(0.5) > 0 \Rightarrow (0, -4)$ is a minimum

$f'(0.5) > 0; f'(1.5) < 0 \Rightarrow (1, -2)$ is a maximum

ii $\frac{dy}{dx} > 0$ when $x < -\frac{1}{2}$ or $0 < x < 1$

iii $\frac{dy}{dx} < 0$ when $-\frac{1}{2} < x < 0$ or $x > 1$

iv $\left(-\frac{1}{2}, -\frac{59}{16}\right)$ and (0,-4) are local turning points, and (1,-2) is a global maximum.

c $y = 2\left(3 - 2x - x^2\right)^{-1}$

i $\therefore \frac{dy}{dx} = -2(-2-2x)\left(3-2x-x^2\right)^{-2} = 4(1+x)\left(3-2x-x^2\right)^{-2}$

$\frac{dy}{dx} = 0$ when $x = -1$

By testing points either side of $x = -1$ in $\frac{dy}{dx}$, $\left(-1, \frac{1}{2}\right)$ is a minimum

ii $\frac{dy}{dx} > 0$ when $x > -1$ $(x \neq 1)$

iii $\frac{dy}{dx} < 0$ when $x < -1$ $(x \neq -3)$

iv $\left(-1, \frac{1}{2}\right)$ is a local minimum.

d $y = \frac{3x+3}{3x-x^2}$

i □ $\frac{dy}{dx} = \frac{\left(3x-x^2\right)(3) - (3x+3)(3-2x)}{\left(3x-x^2\right)^2}$

$= \frac{3x^2+6x-9}{\left(3x-x^2\right)^2} = \frac{3(x-1)(x+3)}{\left(3x-x^2\right)^2}$

$\therefore \frac{dy}{dx} = 0$ when $x = -3$ or $x = 1$

By testing points to the left and right of these x-values in f',

$\left(-3, \frac{1}{3}\right)$ is a maximum and

$(1, 3)$ is a minimum

ii $\frac{dy}{dx} > 0$ when $x < -3$ or $x > 1$ $(x \neq 3)$

iii $\frac{dy}{dx} < 0$ when $-3 < x < 1$ $(x \neq 0)$

iv Both turning points are local.

e $y = x + (x-1)^{-1}$

i $\dfrac{dy}{dx} = 1 - (x-1)^{-2} = 1 - \dfrac{1}{(x-1)^2} = \dfrac{(x-1)^2 - 1}{(x-1)^2}$

$= \dfrac{x^2 - 2x}{(x-1)^2} = \dfrac{x(x-2)}{(x-1)^2}$

so $\dfrac{dy}{dx} = 0$ when $x = 0$ or $x = 2$

By testing points to the left and to the right of these values in f',

$(0, -1)$ is a maximum and

$(2, 3)$ is a minimum

ii $\dfrac{dy}{dx} > 0$ when $x < 0$ or $x > 2$

iii $\dfrac{dy}{dx} < 0$ when $0 < x < 2$ $(x \neq 1)$

iv The turning points are local.

4 $f(x) = 4 + 5x - x^2 - x^3$

$f'(x) = 5 - 2x - 3x^2 = 0$

$\Rightarrow 3x^2 + 2x - 5 = (3x+5)(x-1) = 0$

$\Rightarrow x = -\dfrac{5}{3}$ or $x = 1$

$\therefore$ By testing points to the right and left of these values in f' $\left(-\dfrac{5}{3}, -\dfrac{67}{27}\right)$

is a local minimum and $(1, 7)$ is a local maximum.

but $f(-3) = 7$ and $f(3) = -17$ so the local minimum is not

the least value of $f(x)$ in the given interval:

$\therefore f_{\text{min}} = -17$

$f_{\text{max}} = 7$

5 $f(x) = x^2 + \dfrac{4}{x}$

As $x = 0$ is in the interval, the global maximum and minimum are $\pm\infty$.

6 $\dfrac{dy}{dx} = 3x^2 + 2ax = 0 \Rightarrow x(3x + 2a) = 0$

$\Rightarrow x = 0; x = \dfrac{-2a}{3}$

So the turning points are at x=0 and x=4.

$x = 4 \Rightarrow 4 = -\dfrac{2a}{3} \Rightarrow a = -6$

$\therefore 48 + 8a = 0 \Rightarrow a = -6$
$\Rightarrow f(4) = -11 \Rightarrow 4^3 - 6(4^2) + b = -11$
$\Rightarrow b = 21$
$x = 0 \Rightarrow (0, b) = (0, 21)$

7 The graph passes through $(1,1)$

$\therefore 1 + a + b + c = 1 \Rightarrow a + b + c = 0 \quad (1)$
The graph has turning points at $x = -1$ and $x = 3$
$\frac{dy}{dx} = 3x^2 + 2ax + b$
$\therefore 3 - 2a + b = 0 \Rightarrow 2a - b = 3 \quad (2)$
and

$27 + 6a + b = 0 \Rightarrow 6a + b = -27 \quad (3)$
Solving (2) and (3) simultaneously,
$a = -3$ and $b = -9$ and therefore
$c = -(a + b) = 12$

Exercise 4N

1 a $\frac{dy}{dx} = 7x^6 - 7 = 0 \Rightarrow x = \pm 1$

$\frac{d^2y}{dx^2} = 42x^5$
$42(1)^5 > 0 \Rightarrow (1, -6)$ is a local minimum
$42(-1)^5 < 0 \Rightarrow (-1, 6)$ is a local maximum

b $\frac{dy}{dx} = 20x^3 - 5x^4 = 0 \Rightarrow x^3(20 - 5x) = 0 \Rightarrow x = 0$ or $x = 4$

$\frac{d^2y}{dx^2} = 60x^2 - 20x^3$
$60(4)^2 - 20(4^3) < 0 \Rightarrow (4, 256)$ is a local maximum

At $x = 0$, the 2nd derivative test is inconclusive, so resort to 1st derivative test by testing points to the left and right of $x = 0$, in the first derivative.

$f'(-0.5) < 0; f'(0.5) > 0 \Rightarrow (0,0)$, hence $(0,0)$ is a minimum.

c $\frac{dy}{dx} = 4 - \frac{1}{x^2} = 0 \Rightarrow x = \pm\frac{1}{2}$

$\frac{d^2y}{dx^2} = \frac{2}{x^3}$
$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{2}} > 0 \Rightarrow \left(\frac{1}{2}, 5\right)$ is local minimum by second derivative test
$\left.\frac{d^2y}{dx^2}\right|_{x=-\frac{1}{2}} < 0 \Rightarrow \left(-\frac{1}{2}, -3\right)$ is a local maximum by second derivative test

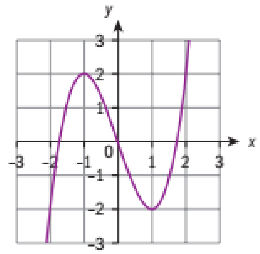
2 a $f'(x) = 3x^2 - b$

$f'(1) = 3 - b = 0 \Rightarrow b = 3$

b $f'(x) = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$

$\therefore (1, -2)$ and $(-1, 2)$

c



3 a $f'(x) = 2ax + b = 0$

$\Rightarrow x = -\dfrac{b}{2a}$

b Local minimum if $a > 0$, local maximum if $a < 0$

Justified as a direct consequence of the second derivative test since $f''(x) = 2a$

4 a $f'(x) = 3x^2 + b$

$f'(1) = 3 + b = 0 \Rightarrow b = -3$
$f(1) = 1 + b + c = c - 2 = 4 \Rightarrow c = 6$

b
$f(x) = x^3 - 3x + 6$
$f'(x) = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$
so there is another turning point at
$(-1, 8)$

Exercise 4O

1 a $f'(x) = 3x^2 - 1$

$f''(x) = 6x$
$f''(0) = 0$ and $y = f(x)$ changes concavity through this point so $(0, 0)$ is a point of inflexion

b $x > 0$

c $x < 0$

2 a $f'(x) = 4x^3 - 3$

$f''(x) = 12x^2$

$f''(0) = 0$ but $y = f(x)$ does not change concavity through this point so no points of inflexion

b $x \in \mathbb{R} \setminus \{0\}$

c $\varnothing$

3 a $f'(x) = 3x^2 - 12x - 12$

$f''(x) = 6x - 12 = 0 \Rightarrow x = 2$

$y = f(x)$ changes concavity through this point so

$(2, -38)$ is a point of inflexion

b $x > 2$

c $x < 2$

4 a $f'(x) = 3x^2 + 2x$

$f''(x) = 6x + 2 = 0 \Rightarrow x = -\frac{1}{3}$

$y = f(x)$ changes concavity through this point so

$\left(-\frac{1}{3}, -\frac{25}{27}\right)$ is a point of inflexion

b $x > -\frac{1}{3}$

c $x < -\frac{1}{3}$

5 a $f'(x) = 12x^2 - 4x^3$

$f''(x) = 24x - 12x^2 = 12x(2 - x) = 0 \Rightarrow x = 0$ or $x = 2$

The concavity of $y = f(x)$ changes through both of these points so

$(0, 0)$ and $(2, 16)$ are both points of inflexion

b $0 < x < 2$

c $x < 0$ or $x > 2$

6 a $f'(x) = 3x^2 - 6x + 3$

$f''(x) = 6x - 6 = 0 \Rightarrow x = 1$

$y = f(x)$ changes concavity through this point so

$(1, 0)$ is a point of inflexion

b $x > 1$

c $x < 1$

7 a $f'(x) = 8x^3 + 3x^2$

$f''(x) = 24x^2 + 6x = 6x(4x+1) = 0$

$\Rightarrow x = -\frac{1}{4}$ or $x = 0$

The concavity of $y = f(x)$ changes through both of these points

so $\left(-\frac{1}{4}, \frac{127}{128}\right)$ and $(0,1)$ are both points of inflexion

b $x < -\frac{1}{4}$ or $x > 0$

c $-\frac{1}{4} < x < 0$

8 **a** $f'(x) = 4x^3 - 12x^2 + 16$

$f''(x) = 12x^2 - 24x = 12x(x-2) = 0$

$\Rightarrow x = 0$ or $x = 2$

The concavity of $y = f(x)$ changes through both of these points

so $(0,-16)$ and $(2,0)$ are points of inflexion

b $x < 0$ or $x > 2$

c $0 < x < 2$

9 **a** $f'(x) = 3ax^2 + 2bx + c$

since there are two distinct stationary points, the discriminant of this quadratic is positive

$\therefore \Delta = (2b)^2 - 4(3a)(c) > 0$

$\Rightarrow 4b^2 - 12ac > 0$

$\Rightarrow b^2 > 3ac$

b $f''(x) = 6ax + 2b = 0 \Rightarrow x = -\frac{b}{3a}$

$(a \neq 0$ by construction since $f(x)$ is a cubic$)$

$f'(x) = 0 \Rightarrow x = \frac{-2b \pm \sqrt{4b^2 - 4(3a)(c)}}{2(3a)} = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}$

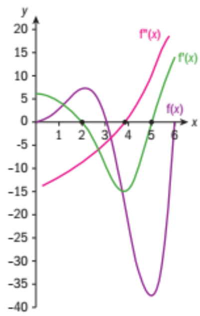
$\therefore$ Let $x_1 = \frac{-b + \sqrt{b^2 - 3ac}}{3a}$ and $x_2 = \frac{-b - \sqrt{b^2 - 3ac}}{3a}$

$\Rightarrow \frac{x_1 + x_2}{2} = \frac{1}{2}\left(\frac{-b + \sqrt{b^2 - 3ac} - b - \sqrt{b^2 - 3ac}}{3a}\right) = -\frac{b}{3a}$

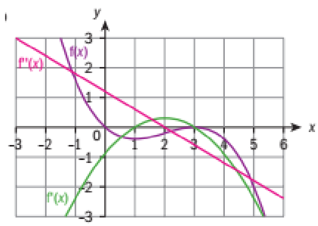
as required

Exercise 4P

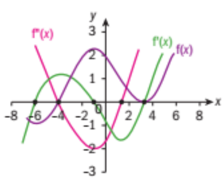
1 a



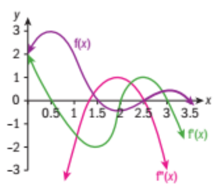
b



c

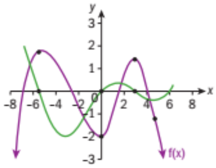


d

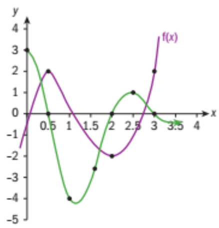


Exercise 4Q

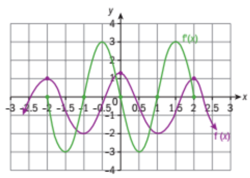
1 a



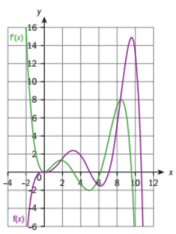
b



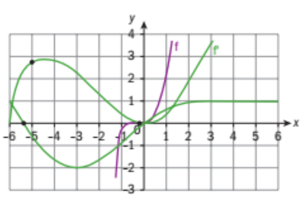
c



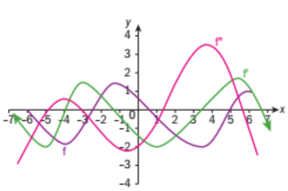
d



2 a



b



Exercise 4R

1 Let the length of the fence opposite the wall be denoted by x

Since there is 800m of fencing available, the remaining sides must each measure $\frac{800-x}{2}$

Therefore the area of the enclosed rectangular plot is

$$A(x) = x\left(\frac{800 - x}{2}\right) = 400x - \frac{x^2}{2}$$

$\therefore A'(x) = 400 - x = 0 \Rightarrow x = 400$

This is a maximum, which can be justified by the shape of the graph or by showing $A''(400) < 0$, indeed $A''(400) = -1$

$$\therefore A_{\min} = A(400) = 400(400) - \frac{400^2}{2} = 80000 \text{ so } 80{,}000 \text{ m}^2$$

2 Let the two side lengths be denoted by x and y, side x being opposite the river

Since the area enclosed is 200m^2, $xy = 200 \Rightarrow y = \dfrac{200}{x}$

and the total length of fencing required is

$$L(x) = x + 2y = x + \frac{400}{x}$$

$$\therefore L'(x) = 1 - \frac{400}{x^2} = 0 \Rightarrow x^2 = 400 \Rightarrow x = 20 \ (x > 0)$$

$L''(20) > 0$ so this is a minimum

$$\therefore L_{\min} = L(20) = 20 + \frac{400}{20} = 40 \text{ so } 40\text{m}$$

3 Let the radius of the semicircle be denoted by r

Then the horizontal edge of the window measures $2r$ and since the perimeter of the window is 12m, if the vertical edges of the window are length x,

$$2r + 2x + \pi r = 12 \Rightarrow x = \frac{12 - (2 + \pi)r}{2}$$

$\therefore$ The area of the window is

$$A(r) = 2xr + \frac{1}{2}\pi r^2 = \left[12 - (2 + \pi)r\right]r + \frac{1}{2}\pi r^2$$

$$= 12r + \left[\frac{\pi}{2} - (2 + \pi)\right]r^2 = 12r - \left(\frac{\pi}{2} + 2\right)r^2$$

$$\therefore A'(r) = 12 - (\pi + 4)r = 0 \Rightarrow r = \frac{12}{4 + \pi}$$

This is a maximum, which can be justified either

by considering the shape of the graph or by computing

$$A''\left(\frac{12}{4 + \pi}\right) = -\pi - 4 < 0$$

$$\therefore r = \frac{12}{4 + \pi} = 1.68\ldots \text{ so radius is } 1.68\text{m} \ (3\text{s.f.})$$

and the dimensions of the rectangle are

$3.36 \text{ m} \times 1.68 \text{ m} \ (3\text{s.f.})$

4 Let L be the length of the wire, then:

$$L = \sqrt{36 + x^2} + \sqrt{196 + (20 - x)^2}$$

$$\frac{dL}{dx} = \frac{x}{\sqrt{36 + x^2}} - \frac{20 - x}{\sqrt{196 + (20 - x^2)}}$$

To minimize the length of L, set the first derivative equal to 0, i.e.,

$$\frac{x}{\sqrt{36+x^2}} - \frac{20-x}{\sqrt{196+(20-x^2)}} = 0$$
$\Rightarrow x = 6$

5 Let the side length of the congruent squares be denoted by x

After cutting out the congruent squares, the sides of the open rectangular box

will measure $(24-2x)$cm by $(45-2x)$cm by x cm

Therefore the volume of the box is

$V(x) = x(24-2x)(45-2x) = 4x^3 - 138x^2 + 1080x$

$\therefore V'(x) = 12x^2 - 276x + 1080 = 0$

$\Rightarrow x^2 - 23x + 90 = 0$

$\Rightarrow (x-18)(x-5) = 0$

$\Rightarrow x = 18$ or $x = 5$

$x = 18$ is not feasible and $V''(5) < 0$ so

$V_{\min} = V(5) = 5(24-10)(45-10) = 5(14)(35) = 2450$

6 Let the height and radius of the cylindrical can respectively be h and r.

Then, since the surface area is 3π m^2,

$$2\pi rh + \pi r^2 = 3\pi \Rightarrow r^2 + 2rh - 3 = 0 \Rightarrow h = \frac{3-r^2}{2r}$$

Therefore the volume of the can is

$$V(r) = \pi r^2 h = \pi r^2 \frac{3-r^2}{2r} = \frac{\pi}{2}(3r - r^3)$$

$$\therefore V'(r) = \frac{\pi}{2}(3-3r^2) = 0 \Rightarrow r = 1 \ \ (r > 0)$$

$\therefore r = 1,\ h = \frac{3-1}{2} = 1$ so the radius and height are both 1m and

$V_{\min} = V(1) = \pi$ so the volume is π m^3

7 $1l = \frac{1}{1000} m^3$

Let the radius of the cylinder be r and the height be h

Then,

$$\pi r^2 h = \frac{1}{1000} \Rightarrow h = \frac{1}{1000\pi r^2}$$

$$S(r) = 2\pi r^2 + 2\pi rh = 2\pi r^2 + \frac{1}{500r}$$

$$S'(r) = 4\pi r - \frac{1}{500r^2} = 0 \Rightarrow r^3 = \frac{1}{2000\pi} \Rightarrow r = \left(\frac{1}{2000\pi}\right)^{\frac{1}{3}}$$

$$\therefore S_{\min} = S\left(\left(\frac{1}{2000\pi}\right)^{\frac{1}{3}}\right) = 0.055358...$$

Therefore the minimum surface area is 0.055358...m^3 or 554cm^3 to 3s.f.

8 Let the radius of the right-circular cone be r. Accordingly, by Pythagoras', the height of the cone is $h = 10 + \sqrt{100 - r^2}$

$\Rightarrow r^2 = 100 - (h - 10)^2 = 20h - h^2$

$\therefore V(h) = \dfrac{\pi r^2 h}{3} = \dfrac{\pi h}{3}(20h - h^2) = \dfrac{\pi}{3}(20h^2 - h^3)$

$\Rightarrow V'(h) = \dfrac{\pi}{3}(40h - 3h^2) = \dfrac{\pi h}{3}(40 - 3h) = 0$

$h \neq 0$ so $h = \dfrac{40}{3}$ and $V''\left(\dfrac{40}{3}\right) < 0$

$\therefore V_{\max} = V\left(\dfrac{40}{3}\right) = 1241.12....$

so 1240cm^3 (3s.f.)

9 A general point on the curve has coordinates $(x, \sqrt{x})$

By Pythagoras', the distance from $(1.5, 0)$ is

$D(x) = \sqrt{\left(x - \dfrac{3}{2}\right)^2 + \left(\sqrt{x}\right)^2} = \sqrt{x^2 - 2x + \dfrac{9}{4}}$

$= \sqrt{(x-1)^2 + \dfrac{5}{4}} \geq \sqrt{\dfrac{5}{4}} = \dfrac{\sqrt{5}}{2}$

so $D_{\min} = \dfrac{\sqrt{5}}{2}$

10 Let the radii of the circles be r

Then the pieces of wire for each circle measure $2\pi r$ and the piece of wire for the square measures $80 - 4\pi r$ so the square has side length $20 - \pi r$
Therefore the area of each circle is πr^2 and the area of the square is $(20 - \pi r)^2$

So the combined area is

$A(r) = 2\pi r^2 + (20 - \pi r)^2 = (2\pi + \pi^2)r^2 - 40\pi r + 400$

$\therefore A'(r) = (4\pi + 2\pi^2)r - 40\pi = 0 \Rightarrow r = \dfrac{20}{2 + \pi}$

and $A''\left(\dfrac{20}{2+\pi}\right) > 0$ so this is a minimum

11 Let the poster have height h and width w

Then, $hw = 320$
and the total area is

$A(h) = (h + 10)(w + 8) = hw + 8h + 10w + 80 = 400 + 8h + 10w$

$= 400 + 8h + \dfrac{3200}{h}$

$\therefore A'(h) = 8 - \dfrac{3200}{h^2} = 0 \Rightarrow h^2 = 400 \Rightarrow h = 20 \ (h > 0)$

$\therefore$ 16cm by 20cm

12 Suppose I land my boat x km along the shore from the point on the coastline directly opposite my initial position. Then, by Pythagoras', I row $\sqrt{x^2+4}$ km and jog $(6-x)$ km so the total time taken is

$$T(x)=\frac{\sqrt{x^2+4}}{2}+\frac{6-x}{5}$$
$$\therefore T'(x)=\frac{x}{2\sqrt{x^2+4}}-\frac{1}{5}=0$$
$$\Rightarrow 5x=2\sqrt{x^2+4}$$
$$\Rightarrow 25x^2=4\left(x^2+4\right)$$
$$\Rightarrow 21x^2=16\Rightarrow x^2=\frac{16}{21}\Rightarrow x=\frac{4}{\sqrt{21}}=\frac{4\sqrt{21}}{21}\quad(x>0)$$

$\therefore$ I should land my boat 0.873km (to 3s.f.) along the coastline from the point directly opposite my initial position

Exercise 4S

1 $C(x)=0.01x^3-10x+150$

$$\therefore C'(x)=0.03x^2-10=0\Rightarrow x^2=\frac{1000}{3}$$
$$\Rightarrow x=\sqrt{\frac{1000}{3}}\quad(x>0)$$

$C''\left(\sqrt{\frac{1000}{3}}\right)>0$ so this is a minimum and a global minimum

by consideration of the graph $\sqrt{\frac{1000}{3}}=18.257\ldots$ so 18

2 $f(x)=\frac{3}{4}x^2-\frac{1}{8}x^3$

The largest drop in systolic pressure is the maximum of $f(x)$

$$\therefore f'(x)=\frac{3}{2}x-\frac{3}{8}x^2=\frac{3x}{8}(4-x)=0\Rightarrow x=4cm^3$$
$$\therefore f(4)=\frac{3}{4}\left(4^2\right)-\frac{1}{8}\left(4^3\right)=4mmHg$$

3 **a** $7\div 0.002=3500,$ so the domain of $p(x)$ is $0\le x\le 3500$

b $x(7-0.002x)-(500+3x)=0$

$$\Rightarrow 0.002x^2-4x+500=0$$
$$\Rightarrow x^2-2000x+250000=0$$
$$x=1000\pm 500\sqrt{3}=1866\ldots \text{ or } 133.97\ldots$$

c From part **b**, $1000-500\sqrt{3}<x<1000+500\sqrt{3}$ or $134<x<1870$

d $3500^2-2000(3500)+250000=5500000$
$=5.5$ million

4 **a** $c'(x)=20-0.4x+0.0012x^2=0$

$\Rightarrow x = 61.25741133$ or $x = 272.075922$
The latter is the minimum so 272

b $p(x) = r(x) - c(x) = 15x - 403 + 0.2x^2 - 0.0004x^3$

$p'(x) = 15 + 0.4x - 0.0012x^2 = 0$
$\Rightarrow x = 367.3599...$ so 367

c Individual Response

5 Let the plot measure x m by y m and suppose that

the side with fencing is of length x
$\therefore xy = 1000$
and the total cost is

$$T(x) = (15+3)x + (15+15)y = 18x + \frac{30000}{x}$$

$$\therefore T'(x) = 18 - \frac{30000}{x^2} = 0$$

$$\Rightarrow x^2 = \frac{30000}{18} \Rightarrow x = \frac{50\sqrt{6}}{3} = 40.8248...$$

$\Rightarrow y = 10\sqrt{6} = 24.4949...$
so the plot is 24.5m by 40.8m (to 3s.f.)
and the minimised cost is

$$T\left(\frac{50\sqrt{6}}{3}\right) = 1470 \ \text{(to 3s.f.)}$$

6 Suppose the airline reduces the price by €10 n times

Then the price of an individual ticket is $500 - 10n$ and the number of passengers is $180 + 2n$, and therefore the revenue is
$r(n) = (500-10n)(180+2n) = 90000 - 800n - 20n^2$

$\therefore r'(n) = -800 - 40n = 0 \Rightarrow n = 20$
so the optimal price is \$300 and the corresponding number of passengers is 220

Exercise 4T

1 a $v(t) = h'(t) = 96 - 32t \Rightarrow v(0) = 96\text{ms}^{-1}$

b $h'(t) = 0 \Rightarrow t = 3\text{s}$

$h_{max} = h(3) = 256\text{ms}^{-1}$

c $h(t) = 16(7 + 6t - t^2) = 16(1+t)(7-t) = 0$

$t > 0 \Rightarrow t = 7$ so $v(7) = -128\text{ms}^{-1}$

2 a $s(0) = 10$

b $s(t) = 5(2 + t - t^2) = 5(1+t)(2-t) = 0$

$t > 0 \Rightarrow t = 2$

c $v(t) = s'(t) = 5 - 10t$ so $v(2) = -15$

$a(t) = v'(t) = s''(t) = -10$

The diver hits the water with a velocity of 15 m s^{-1}, and a constant vertical acceleration of -10 m s^{-2}, which is approximately the force of gravity. Since both velocity and acceleration are negative, the diver is speeding up as he/she approaches the water.

3 Take the ground to have height 0, so that $h_0 = 0$

$\therefore h(t) = 50t - 4.9t^2$

$h'(t) = 50 - 9.8t = 0 \Rightarrow t = \dfrac{50}{9.8}$

$\therefore h_{max} = h\left(\dfrac{50}{9.8}\right) = 127.55... = 128$ to 3s.f.

so the maximum height of the rocket above ground level is 128m

$h(t) = 0$ when $t = 0$ or $t = \dfrac{50}{4.9} = 10.2041...$

so the rocket is at ground level again after 10.2s (to 3s.f.)

4 **a** $t = 0,\ 3,\ 6,\ 11$

b **i** $0 < t < 3,\ 6 < t < 11$

ii $3 < t < 6$

c **i** $t = 1.5$

ii $t = 4.5$

d $t = 1.5,\ 4.5$

e **i** $0 < t < 1.5, 3 < t < 4.5,\ 6 < t < 9$

ii $1.5 < t < 3,\ 4.5 < t < 6,\ 9 < t < 11$

5 $v(t) = s'(t) = -3t^2 - 6t + 4$

$a(t) = v'(t) = s''(t) = -6t - 6$

In the interval $0 < t < 0.528$, velocity and acceleration have different signs, hence the particle is slowing down. At $t = 0.528$ it comes to a stop, and for $0.528 < t < 1$, both velocity and acceleration have the same sign, hence the particle is speeding up.

6 **a** $v_{avg} = \dfrac{s(3) - s(0)}{3} = \dfrac{63}{3} = 21\text{ms}^{-1}$

b $v(t) = s'(t) = 20t - 3t^2$

$\Rightarrow v(3) = 33\text{ms}^{-1}$

$a(t) = v'(t) = s''(t) = 20 - 6t$

$\Rightarrow a(3) = 2\text{ms}^{-2}$

c Both the velocity and accleration are positive, so the particle is speeding up

d $v(t) = 0 \Rightarrow t = 0$ or $t = \frac{20}{3}$

7 **a** $v(t) = s'(t) = 3t^2 - 14t + 11$

$a(t) = v'(t) = s''(t) = 6t - 14$

b $v(t) = 0 \Rightarrow 3t^2 - 14t + 11 = (3t - 11)(t - 1) = 0$

$\therefore t = 1$ and $t = \frac{11}{3}$

c **i** Require $v(t)$ and $a(t)$ to have the same sign

$a(t) > 0$ when $t > \frac{14}{6} = \frac{7}{3}$ and $a(t) < 0$ when $t < \frac{7}{3}$

$v(t) > 0$ when $0 < t < 1$ or $t > \frac{11}{3}$ and $v(t) < 0$ when $1 < t < \frac{11}{3}$

Therefore the particle is speeding up when

$1 < t < \frac{7}{3}$ or $t > \frac{11}{3}$

ii The particle is slowing down when $a(t)$ and $v(t)$ have different signs

Using the working in part i, this is when

$0 < t < 1$ or $\frac{7}{3} < t < \frac{11}{3}$

d $v(t) = 0 \Rightarrow t = 1$ or $t = \frac{11}{3}$

e $D = |s(1) - s(0)| + |s(3) - s(1)| = 5 + 8 = 13$

Exercise 4U

1 **a** $4y\frac{dy}{dx} - 6x = 0 \Rightarrow \frac{dy}{dx} = \frac{3x}{2y}$

b $3y^2\frac{dy}{dx} = 4x^3 \Rightarrow \frac{dy}{dx} = \frac{4x^3}{3y^2}$

c $4x + 8y\frac{dy}{dx} - 3 + 4\frac{dy}{dx} = 0$

$\Rightarrow (8y + 4)\frac{dy}{dx} = 3 - 4x$

$\Rightarrow \frac{dy}{dx} = \frac{3 - 4x}{8y + 4}$

d $2x - 3y - 3x\frac{dy}{dx} + 4y\frac{dy}{dx} = 0$

$$\Rightarrow (4y-3x)\frac{dy}{dx} = 3y-2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{3y-2x}{4y-3x}$$

e $2(x+y)\left(1+\frac{dy}{dx}\right) = -3\frac{dy}{dx}$

$$\left[2(x+y)+3\right]\frac{dy}{dx} = -2(x+y)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2(x+y)}{2(x+y)+3}$$

f $2x^2 = \frac{x+y}{x-y}$

$$4x = \frac{(x-y)\left(1+\frac{dy}{dx}\right)-(x+y)\left(1-\frac{dy}{dx}\right)}{(x-y)^2} = \frac{2x\frac{dy}{dx}-2y}{(x-y)^2}$$

$$\Rightarrow 4x(x-y)^2+2y = 2x\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{4x(x-y)^2+2y}{2x} = \frac{2x(x-y)^2+y}{x}$$

g $1 = \frac{1}{2}\left(4x-18y^2\frac{dy}{dx}\right)(2x^2-6y^3)^{-\frac{1}{2}}$

$$\Rightarrow 4x-18y^2\frac{dy}{dx} = 2\sqrt{2x^2-6y^3}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{18y^2}\left[2\sqrt{2x^2-6y^3}-4x\right]$$

2 $-\frac{3}{x^4}-\frac{3}{y^4}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y^4}{x^4}$

$\therefore$ At $(1,1)$, the gradient of the tangent is -1 and the gradient of the normal is 1

Tangent: $y-1 = -1(x-1) \Rightarrow y = -x+2$

Normal: $y-1 = 1(x-1) \Rightarrow y = x$

3 $-\frac{1}{(x+1)^2}-\frac{1}{(y+1)^2}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{(y+1)^2}{(x+1)^2}$

$\therefore$ At $(1,1)$, the gradient of the tangent is -1 and the gradient of the normal is 1

Tangent: $y-1 = -(x-1) \Rightarrow y = -x+2$

Normal: $y-1 = x-1 \Rightarrow y = x$

4 $x^2+y^2 = 6x+8y$

$$2x + 2y\frac{dy}{dx} = 6 + 8\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}(2y - 8) = 6 - 2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{3 - x}{y - 4}$$

$$\therefore \frac{dy}{dx} = 0 \text{ when } x = 3$$

$$\Rightarrow 9 + y^2 = 18 + 8y$$

$$\Rightarrow y^2 - 8y - 9 = (y - 9)(y + 1) = 0$$

Therefore $(3,9)$ and $(3,-1)$

5 **a** $x + y = x^2 - 2xy + y^2$

$$1 + \frac{dy}{dx} = 2x - 2x\frac{dy}{dx} - 2y + 2y\frac{dy}{dx}$$

$$\Rightarrow (1 + 2x - 2y)\frac{dy}{dx} = 2x - 2y - 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x - 2y - 1}{2x - 2y + 1}$$

b $1 - \frac{dy}{dx} = 1 - \frac{2x - 2y - 1}{1 + 2x - 2y} = \frac{1 + 2x - 2y - (2x - 2y - 1)}{2x - 2y + 1}$

$$= \frac{1 + 2x - 2y - 2x + 2y + 1}{1 + 2x - 2y}$$

$$= \frac{2}{2x - 2y + 1}$$

c Differentiating the result in part *b*,

$$-\frac{d^2y}{dx^2} = \frac{-2\left(2 - 2\frac{dy}{dx}\right)}{(2x - 2y + 1)^2} = -\frac{4\left(1 - \frac{dy}{dx}\right)}{(2x - 2y + 1)^2}$$

$$= -\frac{4\left(\frac{2}{2x - 2y + 1}\right)}{(2x - 2y + 1)^2} = -\frac{8}{(2x - 2y + 1)^3}$$

$$\therefore \frac{d^2y}{dx^2} = \left(\frac{2}{2x - 2y + 1}\right)^3 = \left(1 - \frac{dy}{dx}\right)^3$$

Exercise 4V

1 **a** $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$

b $\frac{dA}{dt} = 2\pi\frac{dr}{dt} + 2\pi\frac{dh}{dt}$

c $\frac{dV}{dt} = \frac{\pi}{3}\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right) = \frac{\pi r}{3}\left(2h\frac{dr}{dt} + r\frac{dh}{dt}\right)$

d $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$

2 Let the diagonal distance be D. Then,

$D^2 = l^2 + w^2 + h^2$

$$\therefore 2D\frac{dD}{dt} = 2l\frac{dl}{dt} + 2w\frac{dw}{dt} + 2h\frac{dh}{dt}$$

$$\Rightarrow \frac{dD}{dt} = \frac{l\frac{dl}{dt} + w\frac{dw}{dt} + h\frac{dh}{dt}}{D}$$

$$= \frac{l\frac{dl}{dt} + w\frac{dw}{dt} + h\frac{dh}{dt}}{\sqrt{l^2 + w^2 + h^2}}$$

3 $\frac{dl}{dt} = 3$, $\frac{dw}{dt} = -3$

a $A = lw \Rightarrow \frac{dA}{dt} = \frac{dl}{dt}w + l\frac{dw}{dt} = 3(w - l)$

$\therefore$ Initially, $\frac{dA}{dt} = 3(7 - 24) = -51$ cm s^{-2}

b $P = 2(l + w) \Rightarrow \frac{dP}{dt} = 2\left(\frac{dl}{dt} + \frac{dw}{dt}\right) = 0$ m s^{-1}

c Let D denote the diagonal distance. Then,

$$D^2 = l^2 + w^2 \Rightarrow 2D\frac{dD}{dt} = 2l\frac{dl}{dt} + 2w\frac{dw}{dt}$$

$$\Rightarrow \frac{dD}{dt} = \frac{l\frac{dl}{dt} + w\frac{dw}{dt}}{\sqrt{l^2 + w^2}} = \frac{3l - 3w}{\sqrt{l^2 + w^2}}$$

$\therefore$ Initially,

$$\frac{dD}{dt} = \frac{3(24 - 7)}{\sqrt{24^2 + 7^2}} = \frac{51}{25} = 2.04 \text{ cm s}^{-1}$$

4 Let the distance of the base of the ladder from the wall be x

and the vertical height of the ladder up the wall be y

Then, by Pythagoras,

$y = \sqrt{100 - x^2}$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{\sqrt{100 - x^2}}\frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt} = -\frac{\sqrt{100 - x^2}}{x}\frac{dy}{dx} = \frac{\sqrt{100 - x^2}}{2x}$$

$\therefore$ when $x = 6$,

$$\frac{dx}{dt} = \frac{\sqrt{100 - 36}}{12} = \frac{2}{3}$$

5 Let the cube have side length l

$V = l^3 \Rightarrow \frac{dV}{dt} = 3l^2 \frac{dl}{dt} = \frac{3}{2}$

$S = 6l^2$

$\therefore \frac{dS}{dt} = 12l \frac{dl}{dt}$

When $V = 27$, $l = 3$ and $\frac{dl}{dt} = \frac{1}{18}$

so $\frac{dS}{dt} = 12(3)\left(\frac{1}{18}\right) = 2$

6 Let the radius of the cylinder be r and the height h

$\frac{dr}{dt} = -3$ and $\frac{dh}{dt} = 6$

$S = 2\pi r^2 + 2\pi rh = 2\pi r(r+h)$

$\therefore \frac{dS}{dt} = 2\pi\left[\frac{dr}{dt}(r+h) + r\left(\frac{dr}{dt} + \frac{dh}{dt}\right)\right]$

$= 2\pi[-3(r+h) + 3r]$

so when $r = 12,\ h = 10$

$\frac{dS}{dt} = 2\pi[-3(22) + 36] = -60\pi$

7 $h^2 = a^2 + b^2$

$2h\frac{dh}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt}$

$\Rightarrow \frac{dh}{dt} = \frac{a\frac{da}{dt} + b\frac{db}{dt}}{\sqrt{a^2+b^2}} = \frac{5a-4b}{\sqrt{a^2+b^2}}$

$\therefore$ When $a = 15$ and $b = 20$,

$\frac{dh}{dt} = \frac{75-80}{\sqrt{225+400}} = -\frac{5}{25} = -\frac{1}{5}$

8 $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$\therefore -7 = 4\pi r^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = -\frac{7}{4\pi r^2}$

When $V = 36\pi$,

$\frac{4}{3}\pi r^3 = 36\pi \Rightarrow r^3 = 27 \Rightarrow r = 3$

so $\frac{dr}{dt} = -\frac{7}{36\pi}$

9 $V = \frac{1}{3}\pi r^2 h \Rightarrow \frac{dV}{dt} = \frac{\pi r}{3}\left(2h\frac{dr}{dt} + r\frac{dh}{dt}\right) = 3$

Let θ denote the angle between the axis of symmetry through the centre of the cone and the curved surface of the cone. Then,

$$\tan\theta = \frac{r}{h} = \frac{1.5}{2} = \frac{3}{4} \Rightarrow \frac{3}{4}h = r \Rightarrow \frac{dr}{dt} = \frac{3}{4}\frac{dh}{dt}$$

$$\therefore \frac{\pi r}{3}\left(\frac{3h}{2}\frac{dh}{dt} + r\frac{dh}{dt}\right) = 3$$

$$\Rightarrow \frac{dh}{dt} = \frac{18}{\pi r(3h+2r)}$$

so when $h = 2$ and $r = 1.5$,

$$\frac{dh}{dt} = \frac{4}{3\pi}$$

10 $A = \pi r^2 \Rightarrow \dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt} \Rightarrow \dfrac{dr}{dt} = \dfrac{1}{2\pi r}\dfrac{dA}{dt} = \dfrac{1}{\pi r}$

$\therefore$ When $r = 5$,

$$\frac{dr}{dt} = \frac{1}{5\pi}$$

Chapter review

1 a Limit does not exist since $\lim\limits_{x\to 1^+} f(x) \neq \lim\limits_{x\to 1^-} f(x)$

b $\lim\limits_{x\to -1} f(x) = 2$

c $\lim\limits_{x\to 0} f(x) = 2.25$

d no limit as there is a cusp at x= 3.

2 a $\lim\limits_{x\to 2}\left(-x^2 + 5x - 2\right) = 4$

b $\lim\limits_{x\to 2}\dfrac{x+3}{x+6} = \dfrac{5}{8}$

c $\lim\limits_{x\to 0}(8-2x)^{\frac{1}{3}} = 2$

d $\lim\limits_{x\to 3}\dfrac{x^2-4x+3}{x-3} = \lim\limits_{x\to 3}\dfrac{(x-3)(x-1)}{x-3} = \lim\limits_{x\to 3}(x-1) = 2$

e $\lim\limits_{x\to\infty}\dfrac{\sqrt{x^2-2}}{x} = \lim\limits_{x\to\infty}\sqrt{1-\dfrac{2}{x^2}} = 1$

f $\lim\limits_{x\to -\infty}\dfrac{2}{x^3+1} = \lim\limits_{x\to -\infty}\dfrac{\dfrac{2}{x^3}}{1+\dfrac{1}{x^3}} = 0$

3 $\lim\limits_{x\to -2^-} f(x) = (-2)^2 - 4 = 0$

$\lim_{x\to-2^+} f(x) = (-2)^3 - 6(-2) = 4$

so not continuous at $x = -2$

4 **a** $\lim_{x\to-1^-} f(x) = \lim_{x\to-1^+} f(x)$

$\Rightarrow 2 + a = 1 \Rightarrow a = -1$

b $\lim_{x\to1^+} f(x) = \lim_{x\to1^-} f(x)$

$\Rightarrow 6 = a - 1 \Rightarrow a = 7$

5 **a** $\lim_{n\to\infty} \frac{n+3}{3n-4} = \lim_{n\to\infty} \frac{1+\frac{3}{n}}{3-\frac{4}{n}} = \frac{1}{3}$

b $\lim_{n\to\infty} \frac{4-2n^2}{n^3-1} = \lim_{n\to\infty} \frac{\frac{4}{n^3}-\frac{2}{n}}{1-\frac{1}{n^3}} = 0$

c No limit

6 $\left|-\frac{1}{3}\right| < 1$ so the series does converge

$S_\infty = 2 \cdot \frac{1}{1-\left(-\frac{1}{3}\right)} = \frac{3}{2}$

7 The common ratio is $\frac{1}{n^2+1}$, so the condition is

$\frac{1}{n^2+1} < 1 \Rightarrow n^2 + 1 > 1 \Rightarrow n^2 > 0$

i.e. the series converges for all non-zero n

$S_\infty = \frac{n^2}{1-\frac{1}{n^2+1}} = \frac{n^2\left(n^2+1\right)}{n^2+1-1} = \frac{n^2\left(n^2+1\right)}{n^2} = n^2 + 1$

8 **a** Vertical: $x^2 - 9 = 0 \Rightarrow x = \pm 3$

Horizontal:

$\lim_{x\to\infty} f(x) = 6$ so $y = 6$

b Vertical: $3x^3 + 81 = 0 \Rightarrow x^3 = -27 \Rightarrow x = -3$

Horizontal:

$\lim_{x\to\infty} f(x) = 0$ so $y = 0$

c Vertical: $x = 0$

$y = x + 1$

9 **a** $y = (1-2x)^5(3x-2)^6$

$$\frac{dy}{dx} = -10(1-2x)^4(3x-2)^6 + (1-2x)^5 \cdot 18(3x-2)^5$$

$$= (1-2x)^4(3x-2)^5\left[-10(3x-2)+18(1-2x)\right]$$

$$= (1-2x)^4(3x-2)^5(-66x+38)$$

$$= -2(33x-19)(1-2x)^4(3x-2)^5$$

b $$\frac{dy}{dx} = \frac{\sqrt{x-1}(2x)-(x^2-1)\frac{1}{2\sqrt{x-1}}}{x-1}$$

$$= \frac{4x(x-1)-(x^2-1)}{2(x-1)^{\frac{3}{2}}} = \frac{3x^2-4x+1}{2(x-1)^{\frac{3}{2}}}$$

c $$\frac{dy}{dx} = \frac{1}{2}\left(2+\frac{3x^2}{2\sqrt{x^3+1}}\right)\frac{1}{\sqrt{2x+\sqrt{x^3+1}}}$$

10 a Vertical: $x^2-1=0 \Rightarrow x=\pm1$

Horizontal:

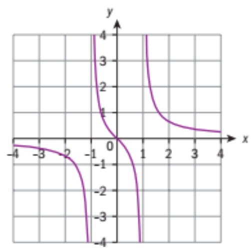
$$\lim_{x\to\infty}\frac{x}{x^2-1} = \lim_{x\to\infty}\frac{\frac{1}{x}}{1-\frac{1}{x^2}} = 0 \quad \text{so} \quad y=0$$

b $$f(x) = \frac{x}{x^2-1}$$

$$f(-x) = \frac{-x}{(-x)^2-1} = -\frac{x}{x^2-1} = -f(x)$$

c $$\frac{dy}{dx} = \frac{(x^2-1)(1)-x(2x)}{(x^2-1)^2} = \frac{-1-x^2}{(x^2-1)^2} = -\frac{1+x^2}{(x^2-1)^2} < 0$$

d



11 a $|OP| = \sqrt{a^2+(3-a^2)^2} = \sqrt{a^4-5a^2+9}$

b $$|OP| = \sqrt{\left(a^2-\frac{5}{2}\right)^2+9-\frac{25}{4}} = \sqrt{\left(a^2-\frac{5}{2}\right)^2+\frac{11}{4}}$$

$\therefore$ Closest when $a^2-\frac{5}{2}=0 \Rightarrow a=\pm\sqrt{\frac{5}{2}}$

12 $f(x)=\dfrac{x^2-3x+2}{x^2+3x+2}=\dfrac{(x-2)(x-1)}{(x+2)(x+1)}$

a Vertical: $x=-2$ and $x=-1$

Horizontal:

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{1-\frac{3}{x}+\frac{2}{x^2}}{1+\frac{3}{x}+\frac{2}{x^2}}=1 \quad \text{so} \quad y=1$$

b $(x-2)(x-1)=0\Rightarrow x=2$ or $x=1$

so $(2,0)$, $(1,0)$, $(0,1)$

c $f'(x)=\dfrac{(x^2+3x+2)(2x-3)-(x^2-3x+2)(2x+3)}{(x^2+3x+2)^2}$

$$=\frac{2x^3+3x^2-5x-6-2x^3+3x^2+5x-6}{(x^2+3x+2)^2}$$

$$=\frac{6x^2-12}{(x^2+3x+2)^2}=0\Rightarrow x=\pm\sqrt{2}$$

13 a $f(x)=0\Rightarrow x=-1$

b Horizontal Asymptote: $y=0$

Vertical Asymptote: $x=0$

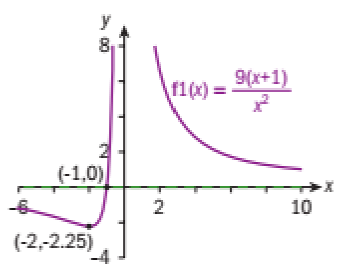
c $f'(x)=0\Rightarrow x=-2$

When $x=-2$, $y=f(-2)=-\dfrac{9}{4}$

$f''(-2)=\dfrac{9}{8}>0$ so $\left(-2,-\dfrac{9}{4}\right)$ is a minimum

d $f''(x)>0\Rightarrow x>-3$

e



14 a $f(x)=0\Rightarrow\sqrt{x}\left(\sqrt{x}-b\right)=0\Rightarrow x=0$ or $x=b^2$

b $f'(x)=1-\dfrac{b}{2\sqrt{x}}=\dfrac{2\sqrt{x}-b}{2\sqrt{x}}$

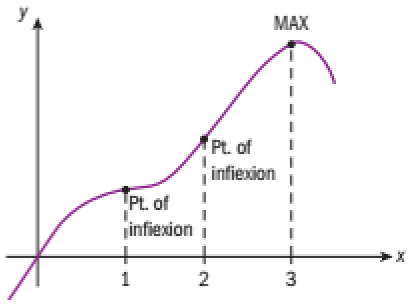
i $f'(x) > 0$ when $x > \dfrac{b^2}{4}$

ii $f'(x) < 0$ when $0 \le x < \dfrac{b^2}{4}$

c $f(x) \ge f\left(\dfrac{b^2}{4}\right) = -\dfrac{b^2}{4}$

d $f''(x) = \dfrac{b}{4x\sqrt{x}} > 0$ so concave up

15



16 $8x + 2y\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{4x}{y}$

$\therefore$ At $\left(\dfrac{2}{\sqrt{5}}, \dfrac{2}{\sqrt{5}}\right)$,

$\dfrac{dy}{dx} = -4$

17 $4xy + 2x^2\dfrac{dy}{dx} + 3y^2 + 6xy\dfrac{dy}{dx} = 0$

$\Rightarrow (2x^2 + 6xy)\dfrac{dy}{dx} = -(3y^2 + 4xy)$

$\Rightarrow \dfrac{dy}{dx} = -\dfrac{3y^2 + 4xy}{2x^2 + 6xy}$

$\therefore$ At $(1,1)$,

$\dfrac{dy}{dx} = -\dfrac{3+4}{2+6} = -\dfrac{7}{8}$

so the normal to the curve at this point has gradient $\dfrac{8}{7}$

$\therefore y - 1 = \dfrac{8}{7}(x - 1)$

$\Rightarrow y = \dfrac{8}{7}x - \dfrac{1}{7}$

18 $\sqrt{a}+\sqrt{b}=\sqrt{p}$

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\sqrt{\frac{y}{x}}$$

$\therefore$ The gradient at (a,b) is $-\sqrt{\frac{b}{a}}$

and the equation of the tangent is

$$y-b=-\sqrt{\frac{b}{a}}(x-a)$$

$$\Rightarrow y=-\sqrt{\frac{b}{a}}x+\sqrt{ab}+b$$

The tangent passes through $(0,r)$:

$r=\sqrt{ab}+b$

and $(s,0)$:

$$0=-\sqrt{\frac{b}{a}}s+\sqrt{ab}+b \Rightarrow s=a+\sqrt{ab}$$

$\therefore r+s=\sqrt{ab}+b+a+\sqrt{ab}=a+2\sqrt{ab}+b$

$=\left(\sqrt{a}+\sqrt{b}\right)^2=\left(\sqrt{p}\right)^2=p$ as required

19 Let D denote the distance from the observer to the drone and x the height of the drone. Then,

$$D^2=36+x^2 \Rightarrow 2D\frac{dD}{dt}=2x\frac{dx}{dt}$$

$$\Rightarrow \frac{dD}{dt}=\frac{x}{\sqrt{36+x^2}}\frac{dx}{dt}$$

$\therefore$ When $\frac{dx}{dt}=3$ and $x=8$,

$$\frac{dD}{dt}=\frac{8}{\sqrt{36+64}}(3)=2.4$$

20 a $v(0)=-2$

b $v(t)=0$

$\Rightarrow 1+t=\sqrt{4t+9}$

$\Rightarrow (1+t)^2=1+2t+t^2=4t+9$

$\Rightarrow t^2-2t-8=0$

$\Rightarrow (t-4)(t+2)=0$

$t>0$ so $t=4$

c $a(t)=v'(t)=1-\frac{2}{\sqrt{4t+9}}$

$$a(4)=1-\frac{2}{\sqrt{25}}=\frac{3}{5}$$

d i $v(t) < 0$ for $t < 4$ and $v(t) > 0$ for $t > 4$

$a(t) > 0$ for all $t > 0$

$\therefore$ slowing down when $a(t)$ and $v(t)$ have different signs

$\Rightarrow t < 4$

ii $v(t)$ and $a(t)$ have the same sign when $t > 4 \Rightarrow t > 4$

21 Let the (initially) inner circle have radius r_1 and the (initially) outer circle have radius r_2

Then, $\frac{dr_1}{dt} = 1.2$ and $\frac{dr_2}{dt} = 0.8$

The area between the circles is given by

$A = \pi\left(r_2^2 - r_1^2\right)$ so at time t,

$$\Rightarrow \frac{dA}{dt} = 2\pi\left(r_2\frac{dr_2}{dt} - r_1\frac{dr_1}{dt}\right) = 2\pi(0.8 \times 3 - 1 \times 1.2) = \frac{12\pi}{5}$$

22 $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} = 3 \Rightarrow \frac{dr}{dt} = \frac{3}{4\pi r^2}$$

$$\frac{dS}{dt} = 8\pi r\frac{dr}{dt} = 8\pi r\frac{3}{4\pi r^2} = \frac{6}{r}$$

$\therefore$ When $r = 1$,

$$\frac{dS}{dt} = 6$$

23 $|PQ| = x \Rightarrow |QT| = |RS| = x$ since PQT is an equilateral triangle and QRST is a rectangle. Let $|ST| = |QR| = y$, then

$p = 3x + 2y$ and $A = xy + \frac{\sqrt{3}}{4}x^2$ (area of rectangle plus eq. triangle)

$$\therefore y = \frac{p - 3x}{2}$$

$$\Rightarrow A = \frac{x(p - 3x)}{2} + \frac{\sqrt{3}}{4}x^2$$

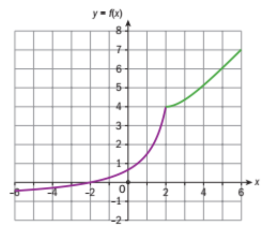
$$\frac{dA}{dx} = \frac{p}{2} - 3x + \frac{\sqrt{3}}{2}x = 0$$

$$\Rightarrow \left(3 - \frac{\sqrt{3}}{2}\right)x = \frac{p}{2}$$

$$\Rightarrow \frac{p}{x} = 2\left(3 - \frac{\sqrt{3}}{2}\right) = 6 - \sqrt{3}$$

Exam-style questions

24 a Graphical approach:



Attempt to draw graph (1 mark)

Each branch correct (2 marks)

Then $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = 4$

Hence $\lim_{x\to 2} f(x)$ exists and is equal to 4. (1 mark)

OR: Algebraic approach (note that an algebraic approach will be accepted, but not expected, in examinations)

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} -\frac{x^2-4}{x^2-5x+6} = \lim_{x\to 2^-} -\frac{(x-2)(x+2)}{(x-2)(x-3)}$$ (1 mark)

$$= \lim_{x\to 2} \frac{-(x+2)}{x-3} = 4$$ (1 mark)

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} e^{2-x} + x + 1 = 4$$ (1 mark)

Then $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = 4$

Hence $\lim_{x\to 2} f(x)$ exists and is equal to 4. (1 mark)

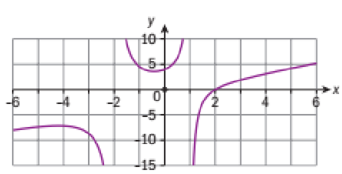
b $a^2 + 3a + 6 = 4$ (1 mark)

$(a+1)(a+2) = 0$ (1 mark)

$a = -1, a = -2$ (1 mark)

25 a $x^2 + x - 2 \neq 0 \Rightarrow x \neq 1, x \neq -2$ (2 marks)

b



(Shape of each branch correct gains 1 mark) (3 marks)

c i Vertical asymptotes: $x = 1$ and $x = -2$ (2 marks)

ii Using long division, $f(x) = x - 1 + \dfrac{3x - 10}{x^2 + x - 2}$ (1 mark)

As $x \to \infty$, $f(x) \to x - 1$ which is a slant asymptote. (1 mark)

26 a $g'(x) = \dfrac{\dfrac{1}{2\sqrt{x}}\left(x^2+1\right) - \sqrt{x} \cdot 2x}{\left(x^2+1\right)^2}$ (2 marks)

$= \dfrac{1 - 3x^2}{2\sqrt{x}\left(x^2+1\right)^2}$ (1 mark)

b $g(1) = \dfrac{1}{2}$ (1 mark)

$g'(1) = -\dfrac{1}{4}$ (1 mark)

Equation of normal: $y - \dfrac{1}{2} = 4(x-1)$ or $y = 4x - 3\dfrac{1}{2}$ (1 mark)

c g' is not defined at $x = 0$ because a derivative is not defined at the end point of a closed interval. (1 mark)

Therefore, there is no tangent to the graph of g at $x = 0$. (1 mark)

27 a $2x = \dfrac{\left(1 + \dfrac{dy}{dx}\right)(x - y) - \left(1 - \dfrac{dy}{dx}\right)(x + y)}{(x-y)^2}$ (3 marks)

Make $\dfrac{dy}{dx}$ the subject (1 mark)

$2x(x-y)^2 = 2x\dfrac{dy}{dx} - 2y$

$\dfrac{dy}{dx} = (x-y)^2 + \dfrac{y}{x}$ (1 mark)

b $y = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$ (1 mark)

$\dfrac{dy}{dx} = 1$ (1 mark)

$y = x \mp 1$ (2 marks)

28 a i $\dfrac{101.1 - 98.5}{2002 - 2000} = 1.3$ (2 marks)

ii $\dfrac{102.3 - 101.1}{2004 - 2002} = 0.6$ (1 mark)

b The average annual profit between 2000 and 2002 was almost double the average annual profit between 2002 and 2004. (2 marks)

29 a $f'(x)=\lim_{h\to 0}\dfrac{\left(2(x+h)^2+3(x+h)-4\right)-\left(2x^2+3x-4\right)}{h}$ (2 marks)

$=\lim_{h\to 0}\dfrac{\left(2x^2+4hx+2h^2+3x+3h-4\right)-\left(2x^2+3x-4\right)}{h}$ (1 mark)

$=\lim_{h\to 0}\dfrac{4hx+2h^2+3h}{h}$ (1 mark)

$=\lim_{h\to 0}\left(4x+2h+3\right)=4x+3$ (1 mark)

b $f(1)=2+3-4=1$ (1 mark)

$f'(1)=4+3=7$ (1 mark)

Equation of tangent: $y-1=7(x-1)$ (or $y=7x-6$) (1 mark)

30 a $h(4)=370$ and $h(5)=438$ (3 s.f.) (2 marks)

b $v(t)=h'(t)=112-9.8t$ (2 marks)

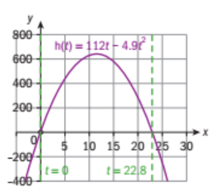
c $v(t)=0\Rightarrow 112-9.8t=0$ (1 mark)

$t=11.4$ (3 *s.f.*) (1 mark)

d double x-coordinate of maximum, or determine zero (1 mark)

22.9 (3 *s.f.*) (1 mark)

e



(Shape: 1 mark; Domain: 1 mark; Maximum: 1 mark)

f $v(22.9)=-112\ \text{ms}^{-1}$ (2 marks)

g $a(t)=v'(t)=-9.8$ which is constant (2 marks)

31 a i $\left(\dfrac{f}{g}\right)'(2)=\dfrac{f'(2)g(2)-f(2)g'(2)}{\left(g(2)\right)^2}$ (1 mark)

$=\dfrac{10\times 4-9\times\left(-\dfrac{4}{3}\right)}{4^2}$ (1 mark)

$=\dfrac{52}{16}\left(=\dfrac{13}{4}=3.25\right)$ (1 mark)

ii $(g\circ f)'(1)=g'(f(1))f'(1)=g'(2)f'(1)$ (1 mark)

$= -\frac{4}{3} \times 4 = -\frac{16}{3}$ (1 mark)

b **i** False (1 mark)

as derivative changes sign. (1 mark)

ii False (1 mark)

as the derivatives at these points are not negative reciprocals. (1 mark)

32 a Let $d(x)$ be the total length of the pipeline.

$d(x) = \sqrt{x^2 + 75^2} + 100 - x$ (2 marks)

b Let $c(x)$ be proportional to the construction costs of the pipeline.

$c(x) = 3\sqrt{x^2 + 75^2} + 100 - x$ (1 mark)

$\frac{\mathrm{d}c}{\mathrm{d}x} = \frac{3x}{\sqrt{x^2 + 75^2}} - 1$ (2 marks)

$\frac{\mathrm{d}c}{\mathrm{d}x} = 0 \Rightarrow \frac{3x}{\sqrt{x^2 + 75^2}} = 1$ (1 mark)

Solve equation

$9x^2 = x^2 + 75^2$ (2 marks)

$x = \frac{75}{4}\sqrt{2}$ (1 mark)

c $d(x) = \sqrt{\left(\frac{75}{4}\sqrt{2}\right)^2 + 75^2} + 100 - \frac{75}{4}\sqrt{2}$ (2 marks)

$= \sqrt{75^2\left(\frac{2}{16} + 1\right)} + 100 - \frac{75}{4}\sqrt{2}$

$= 75\sqrt{\frac{9}{8}} + 100 - \frac{75}{4}\sqrt{2}$ (1 mark)

$= 3 \times \frac{75}{4}\sqrt{2} + 100 - \frac{75}{4}\sqrt{2}$

$= \frac{75}{2}\sqrt{2} + 100$ (1 mark)

33 a Vertical Asymptote: $x = a$ (1 mark)

Horizontal Asymptote: $y = a$ (1 mark)

b $f'(x) = \frac{a \cdot (x-a) - 1 \cdot (ax - 4)}{(x-a)^2} = \frac{4 - a^2}{(x-a)^2}.$ (3 marks)

c $f'(x) = 0$ for turning points (1 mark)

$\frac{4 - a^2}{(x-a)^2} = 0 \Rightarrow a^2 = 4 \Rightarrow a = 2$ (1 mark)

For $a = 2$, $f(x) = \frac{2x-4}{x-2} = \frac{2(x-2)}{x-2} = 2$ so the function is constant, and there are no turning points. (1 mark)

For $a \neq 2$, $f'(x) \neq 0$, so the function has no max/min. (1 mark)

d $f'(1) = \frac{4-a^2}{(1-a)^2}$ (1 mark)

gradient of normal is $m = \frac{(1-a)^2}{a^2-4}$ (1 mark)

$f(1) = \frac{a-4}{1-a}$ (1 mark)

$y - \frac{a-4}{1-a} = \frac{(1-a)^2}{a^2-4}(x-1)$ (or equivalent) (1 mark)

e Asymptotes intersect at (a, a). Substitute (a, a) into normal equation. (1 mark)

$a - \frac{a-4}{1-a} = \frac{(1-a)^2}{a^2-4}(a-1)$ (or equivalent) (1 mark)

Simplify (1 mark)

$(a^2-4)^2 = (a-1)^4$

$4a^3 - 14a^2 + 4a + 15 = 0$ (1 mark)

f From GDC (1 mark)

$a = 2.5$ or $a = 1.82$ (2 marks)

(For part **f**, award 2 marks only if negative root $a = -0.823$ is included)

Paper 3

a

			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

A2

b i 1 **ii** n A1A1

c i 15 **ii** 15 A1A1

d i ${}^nC_{n-r} = \frac{n!}{(n-(n-r))!(n-r)!} = \frac{n!}{(r)!(n-r)!} = {}^nC_r$ M1A1

ii Answer to **i** explains the symmetry about a vertical line down the middle of Pascal's triangle. R1

e i 20 **ii** 20 A1A1

f $^nC_{r-1} + {}^nC_r = \dfrac{n!}{(n-r+1)!(r-1)!} + \dfrac{n!}{(n-r)!r!} = \dfrac{n!(r+n-r+1)}{(n-r+1)!(r)!}$ M1A1

$= \dfrac{n!(n+1)}{((n+1)-r)!r!} = \dfrac{(n+1)!}{((n+1)-r)!r!} = {}^{n+1}C_r$ M1AG

AG stands for 'As Given', so no marks will be given for quoting the result you're asked to show

g i $1+3+3+1=8$ **ii** $1+4+6+4+1=16$ A1A1

iii Based on the answers to part **i**, $\sum_{r=0}^{n} {}^nC_r = 2^n$. R2

h i $(a+b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1}b + {}^nC_2\, a^{n-2}b^2 + {}^nC_3\, a^{n-3}b^4 + \ldots + {}^nC_n\, b^n$ M1A1

ii $(2)^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n$ proving that $\sum_{r=0}^{n} {}^nC_r = 2^n$ M1A1A1

iii $0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots + (-1)^n\left({}^nC_n\right)$ proving that $\sum_{r=0}^{n}(-1)^r\left({}^nC_r\right) = 0$ M1A1A1

j The coefficient of x^n in the expansion of $(1+x)^{2n}$ is ${}^{2n}C_n$. A1

The coefficient of x^n in the expansion of $(1+x)^n(1+x)^n$ is found by multiplying the coefficient of x^r by the coefficient of x^{n-r} and then summing from $r=0$ to $r=n$, that is R1

$\left({}^nC_0\right)\left({}^nC_n\right)+\left({}^nC_1\right)\left({}^nC_{n-1}\right)+\left({}^nC_2\right)\left({}^nC_{n-2}\right)+\ldots+\left({}^nC_n\right)\left({}^nC_0\right)$ M1A1

Equating the two, ${}^{2n}C_n = \left({}^nC_0\right)\left({}^nC_n\right)+\left({}^nC_1\right)\left({}^nC_{n-1}\right)+\left({}^nC_2\right)\left({}^nC_{n-2}\right)+\ldots+\left({}^nC_n\right)\left({}^nC_0\right)$

By part **d**, ${}^nC_{n-r} = {}^nC_r$ so R1

${}^{2n}C_n = \left({}^nC_0\right)^2+\left({}^nC_1\right)^2+\left({}^nC_2\right)^2+\ldots+\left({}^nC_n\right)^2$ AG

5 Analysing data and quantifying randomness: statistics and probability

Skills check

1 a Mean $= \frac{720+750+690+975+700+710+720+680+695+645}{10} = \frac{1457}{2} = 728.5$ kg

The number that occurs most often is 720

Median $= \left(\frac{n+1}{2}\right)^{th} = \left(\frac{11}{2}\right)^{th} = \frac{700+710}{2} = 705$

b Range $= 975 - 645 = 330$ kg

Q_1 is the median of the first half of the list, 690 kg

Q_3 is the median of the second half of the list, 720 kg

$IQR = Q_3 - Q_1 = 720 - 690 = 30$ kg

2 Mean $= \frac{(2.5\times5)+(7.5\times2)+(12.5\times6)+(17.5\times8)+(22.5\times4)+(27.5\times5)+(32.5\times8)}{5+2+6+8+4+5+8} = 19.21$ litres

The data is bimodal and the modal classes are $15 < x \le 20$ and $30 < x \le 35$

The median is the $= \left(\frac{n+1}{2}\right)^{th} = \left(\frac{38+1}{2}\right)^{th}$ value which lies in the interval $15 < x \le 20$

Exercise 5A

1 a The target population is "all celery sticks grown in a certain US state"

b The sampling unit is "each celery stick"

c The sample frame is "a list of all celery sticks from the state"

d The sample variable is "the length of the celery stick"

e The sampling values are "the positive real numbers"

2 a The target population is "all ball bearings manufactured by a company"

b The sampling unit is "each ball bearing"

c The sample frame is "a list of all ball bearings enumerated"

d The sample variable is "the weight of the ball bearing"

e The sampling values are "the positive real numbers"

3 a The target population is "all 1 litre soda bottles from a soft drink factory"

b The sampling unit is "each 1 litre soda bottle"

c The sample frame is "all soda bottles enumerated in a list"

d The sample variable is "the volume of the 1 litre soda bottle"

e The sampling values are "the natural numbers"

4 a The target population is "all crates of 50 oranges"

b The sampling unit is "each crate of 50 oranges"

c The sample frame is "an enumerated list of all crates"

d The sample variable is "the weight of a crate of 50 oranges"

e The sampling values are "the positive real numbers"

Exercise 5B

1 List and enumerate all books, generate a random number x then take books $x, x+50, \ldots$

2 a Generate a random number x and then sample x bases from each region

b individual response

3 a The number of samples of three from $0, 1, 2, 3, 4$ is equal to $\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{120}{6 \times 2} = 10$

b The 10 possible samples are $(0,1,2)$, $(0,1,3)$, $(0,1,4)$, $(0,2,3)$, $(0,2,4)$, $(0,3,4)$, $(1,2,3)$, $(1,2,4)$, $(1,3,4)$, $(2,3,4)$

c The means are:

$$\frac{0+1+2}{3} = 1$$

$$\frac{0+1+3}{3} = \frac{4}{3} = 1.3333$$

$$\frac{0+1+4}{3} = \frac{5}{3} = 1.6667$$

$$\frac{0+2+3}{3} = \frac{5}{3} = 1.6667$$

$$\frac{0+2+4}{3} = 2$$

$$\frac{0+3+4}{3} = \frac{7}{3} = 2.3333$$

$$\frac{1+2+3}{3} = 2$$

$$\frac{1+2+4}{3} = \frac{7}{3} = 2.3333$$

$$\frac{1+3+4}{3} = \frac{8}{3} = 2.6667$$

$$\frac{2+3+4}{3} = 3$$

d Mean of population: $\frac{0+1+2+3+4}{5} = \frac{10}{5} = 2$

Mean of sample means:

$$\frac{1+\frac{4}{3}+\frac{5}{3}+\frac{5}{3}+2+\frac{7}{3}+2+\frac{7}{3}+\frac{8}{3}+3}{10} = \frac{20}{10} = 2$$

4 The variable is "whether the envelope is sealed correctly"
The sample is the "batch of selected envelopes"
The population is "all the envelopes"
The variable is discrete

5 It is not possible to wait 4000 years to see if they will last that long

6 This is stratified sampling

7 Pick 12.5 students from each grade (13 from two and 11 from another two).

Exercise 5C

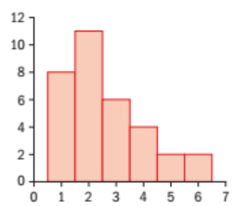
1 Discrete data

a The classes are:
$0.5 < x \le 1.5$
$1.5 < x \le 2.5$
$2.5 < x \le 3.5$
$3.5 < x \le 4.5$
$4.5 < x \le 5.5$
$5.5 < x \le 6.5$

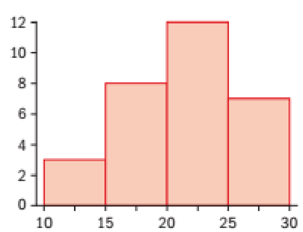
b

Number of people	Frequency	Interval on histogram
1	8	$0.5 < x \le 1.5$
2	11	$1.5 < x \le 2.5$
3	6	$2.5 < x \le 3.5$
4	4	$3.5 < x \le 4.5$
5	2	$4.5 < x \le 5.5$
6	2	$5.5 < x \le 6.5$

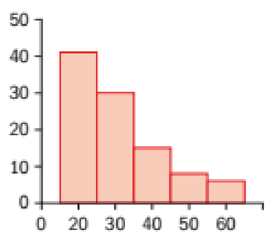
c



2 Continuous data



3 Discrete data



4 **a** The data is continuous

b This plot may show the distribution of lengths of ants in mm

5 Continuous data rounded to discrete

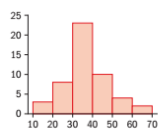
a

Hours	**Days**
4	4
5	5
6	9
7	8
8	4

b $\frac{4\times4+5\times5+6\times9+7\times8+8\times4}{4+5+9+8+4}=\frac{183}{30}=6.1$ hours

Exercise 5D

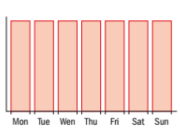
1



Shape: the distribution is unimodal, most koalas had a mass of $30 < x \le 40$ kg
Centre: the midpoint would fall in the $30 < x \le 40$ class.
Spread: the mass of the koalas varies from 17 kg to 61 kg

2 **a** The data is qualitative therefore, a bar chart is preferable. Each bar would represent each day of the week and would summarise the data very clearly

b

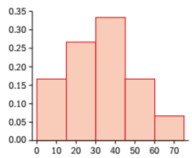


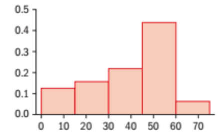
3 **a** A relative frequency histogram is necessary here as we wish to compare the distributions of two samples from different populations

b

Time spent per day	**Male Relative Freq**	**Female Relative Freq**
$0 \le x < 15$	0.1667	0.125
$15 \le x < 30$	0.2667	0.1563
$30 \le x < 45$	0.3333	0.2188
$45 \le x < 60$	0.1667	0.4375
$60 \le x < 75$	0.06667	0.0625

c





d The male distribution is symmetric unimodal. The female distribution is right distorted unimodal.

e On average, females spent more time per day on the phone than men.

4 **a** $8+16+11+7+3=45$ families were interviewed

b $\frac{(150\times 8)+(160\times 16)+(170\times 11)+(180\times 7)+(190\times 3)}{45}=\frac{7460}{45}=\$166.78$

c The data is left skewed

5 $0.25+0.1875+0.125=0.5625$, $32\times 0.5625=18$ items

6 **i** Skewed, bimodal, contains an outlier

ii Skewed, multimodal, no outliers

iii Symmetric, unimodal, no outliers

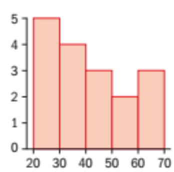
Exercise 5E

1 **a** The mean number of children is $\frac{(0\times 5)+(1\times 10)+(2\times 6)+(3\times 3)+(4\times 1)}{25}=\frac{35}{25}=1.4$

b On average, women from Australia have more children

2 **a** Mean $=\frac{(25\times 5)+(35\times 4)+(45\times 3)+(55\times 2)+(65\times 3)}{5+4+3+2+3}=\frac{705}{17}=41.4706\approx 41.5$ years

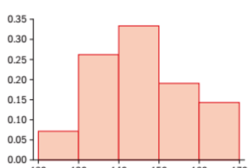
b

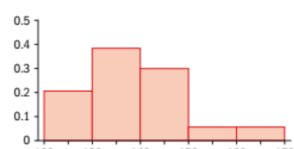


c Left skewed, younger teachers appear to move schools more frequently.

3 **a** A relative frequency histogram is necessary here as we wish to compare the distributions of two samples of different sizes from different populations

b

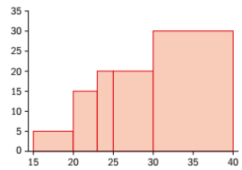




c On average, students from Peru are shorter

4 **a** A standard histogram

b



c Mean $= \dfrac{(17.5\times5)+(21.5\times15)+(24\times20)+(27.5\times20)+(35\times30)}{5+15+20+20+30} = \dfrac{2490}{90} = 27.6667$ years

Modal class: $30 < A \le 40$

Exercise 5F

1 a Mean $= \dfrac{2+3+3+4+4+5+5+6+6+6}{10} = \dfrac{44}{10} = 4.4$

Standard deviation

$$= \sqrt{\frac{2^2+3^2+3^2+4^2+4^2+5^2+5^2+6^2+6^2+6^2}{10} - 4.4^2} = \sqrt{21.2-19.36} = 1.3565 = 1.36$$

b Mean $= \dfrac{21+21+24+25+27+29}{6} = \dfrac{147}{6} = 24.5$ kg

Standard deviation

$$= \sqrt{\frac{21^2+21^2+24^2+25^2+27^2+29^2}{6} - 24.5^2} = \sqrt{608.833-600.25} = 2.92968 = 2.93 \text{ kg}$$

c Mean $= \dfrac{3\times2+4\times3+5\times2}{2+3+2} = \dfrac{28}{7} = 4$

Standard deviation $= \sqrt{\dfrac{3^2\times2+4^2\times3+5^2\times2}{2+3+2} - 4^2} = \sqrt{16.5714-16} = 0.75591 = 0.756$

d Mean $= \dfrac{3\times2+8\times4+13\times4+18\times5+23\times2}{2+4+4+5+2} = \dfrac{226}{17} = 13.2941 \approx 13.3$

Standard deviation

$$= \sqrt{\frac{3^2\times2+8^2\times4+13^2\times4+18^2\times5+23^2\times2}{2+4+4+5+2} - 13.2941^2} = \sqrt{213.412-176.733} \approx 6.06$$

2 Mean $= \dfrac{\Sigma fx}{\Sigma f} = \dfrac{563}{20} = 28.15$

Standard deviation $= \sqrt{\dfrac{\Sigma fx^2}{\Sigma f} - \left(\dfrac{\Sigma fx}{\Sigma f}\right)^2} = \sqrt{\dfrac{16143}{20} - \left(\dfrac{563}{20}\right)^2} = 3.83764 \approx 3.84$

3 a Mean $= \dfrac{196+197+199+200+200+200+202+203+203+205}{10} = \dfrac{2005}{10} = 200.5$ g

Standard deviation

$$= \sqrt{\frac{196^2+197^2+199^2+200^2+200^2+200^2+202^2+203^2+203^2+205^2}{10} - 200.5^2}$$

$$= \sqrt{40207.3-40200.25} = 2.6551 \approx 2.66 \text{ g}$$

4 a $\text{Mean} = \frac{6.3+9.6+12.2+12.3+10.3+12.1+10.3+8.4+9.2+4.3}{10} = \frac{95}{10} = 9.5$

Standard deviation

$$= \sqrt{\frac{6.3^2+9.6^2+12.2^2+12.3^2+10.3^2+12.1^2+10.3^2+8.4^2+9.2^2+4.3^2}{10} - 9.5^2}$$

$$= \sqrt{96.426 - 90.25} \approx 2.49$$

b There is grounds for investigation because the mean amount of lead per litre is within 1 standard deviation of the level that is deemed dangerous

Exercise 5G

1

$$s_{n-1} = \sqrt{\frac{10}{10-1}\left(\frac{10^2+12^2+5^2+0^2+14^2+2^2+5^2+8^2+9^2+6^2}{10} - \left(\frac{10+12+5+0+14+2+5+8+9+6}{10}\right)^2\right)}$$

$$= \sqrt{\frac{10}{9}(67.5-50.41)} = 4.35762 \approx 4.36$$

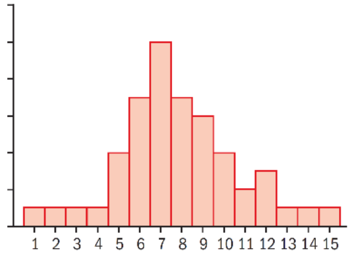
2 $\text{Mean} = \frac{11504}{25} = 460.16 = 460 \text{ kg kg}$

$$s_{n-1} = \sqrt{\frac{25}{25-1}\left(\frac{5304823}{25} - \left(\frac{11504}{25}\right)^2\right)} = 18.72538 = 18.7 \text{ kg}$$

3 $\text{Mean} = \frac{\Sigma x}{n} = \frac{38750}{25} = 1550$

$$s_{n-1} = \sqrt{\frac{n}{n-1}\left(\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2\right)} = \sqrt{\frac{25}{25-1}\left(\frac{60100000}{25} - \left(\frac{38750}{25}\right)^2\right)} = 39.5$$

4 a



b The data is symmetric and unimodal at 7

c $\text{Mean} = \frac{\Sigma x}{n} = \frac{392}{50} = 7.84$

$$s_{n-1} = \sqrt{\frac{50}{50-1}\left(\frac{3470}{50} - \left(\frac{392}{50}\right)^2\right)} = 2.8454 \approx 2.85$$

d Individual response

Exercise 5H

1 a 80

b *a* is lower quartile mark, *a* = 55

b is upper quartile mark, $b = 75$

2 a $\frac{k-3+k+k+2+k+5}{4} = \frac{4k+4}{4} = k+1$

b k + 1 − 3 = k − 2

3 a 63

b i 87 **ii** 73

4 a i 1.18m

ii IQR = UQ − LQ = 1.22m − 1.13m = 0.09m

b

Class	Frequency
$1.00 \le h < 1.05$	5
$1.05 \le h < 1.10$	8
$1.10 \le h < 1.15$	14
$1.15 \le h < 1.20$	24
$1.20 \le h < 1.25$	18
$1.25 \le h < 1.30$	11

c i $\frac{5+8+14}{80} = 0.3375 = 0.34$

ii $\frac{5+8+14\left(\frac{0.02}{0.05}\right)}{5+8+14} = 0.69$

5 a i Mean $= \frac{a+2a+3a+\ldots+na}{n}$

$= \frac{a(1+2+3\ldots+n)}{n} = \frac{a}{n}\frac{n(n+1)}{2} = \frac{a(n+1)}{2}$

ii $\frac{4n(n+1)}{2} > 100 + \frac{4(n+1)}{2}$

$2n^2 + 2n > 100 + 2n + 2$
$2n^2 > 102$
$n^2 > 51$
$n > 7$
$n = 8$

b i $M = \frac{m(0)+n(1)}{n+m} = \frac{n}{n+m}$

$S = \sqrt{\frac{m\times 0^2 + n\times 1^2}{m+n} - \left(\frac{n}{n+m}\right)^2} + \sqrt{\frac{n}{n+m} - \frac{n^2}{(n+m)^2}} = \frac{\sqrt{mn}}{m+n}$

ii $M = S \Rightarrow \frac{n}{n+m} = \frac{\sqrt{nm}}{m+n}$

$n = \sqrt{nm}$

$n^2 = nm$

$n = m$

As there are the same number of x and y points, median is the average of the two values

$\frac{1+0}{2} = 0.5$

Exercise 5I

1 **a** No association

b moderate, positive, linear

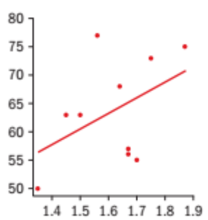
c strong, positive, linear

d moderate, negative, linear

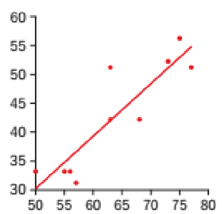
e strong, negative, linear

2

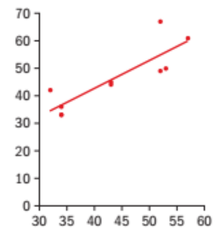
a **i**



b **i**



c **i**

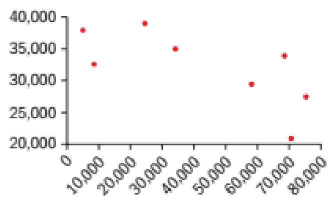


ii 65kg

iii 47s

iv The graphs giving the most accurate predictions are the ones where the data is close to the line of best fit. Graphs b and c are better than graph a.

3



x - Distance (km)	y - Price ($)	x^2	y^2	xy
4895	37900	23961025	1436410000	185520500
75256	27495	5663465536	755975025	2069163720
8563	32595	73324969	1062434025	279110985
24495	38995	600005025	1520610025	955182525
68562	33895	4700747844	1148871025	2323908990
58200	29495	3387240000	869955025	1716609000

34011	34995	1156748121	1224650025	1190214945
70568	21000	4979842624	441000000	1481928000
$\Sigma x = 344550$	$\Sigma y = 256370$	$\Sigma x^2 = 20585335144$	$\Sigma y^2 = 8459905150$	$\Sigma xy = 10201638665$

$$S_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n}$$

$$S_{xx} = 20585335144 - \frac{344550^2}{8} = \frac{11491994663}{2}$$

$$S_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n}$$

$$S_{yy} = 8459905150 - \frac{256370^2}{8} = \frac{488416075}{2}$$

$$S_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}$$

$$S_{xy} = 10201638665 - \frac{344550 \times 256370}{8} = -\frac{1679793545}{2}$$

$$r = \frac{S_{xy}}{\sqrt{(S_{xx}S_{yy})}}$$

$$r = \frac{-\frac{1679793545}{2}}{\sqrt{\frac{11491994663}{2} \times \frac{488416075}{2}}} = -0.709$$

There is moderate negative correlation

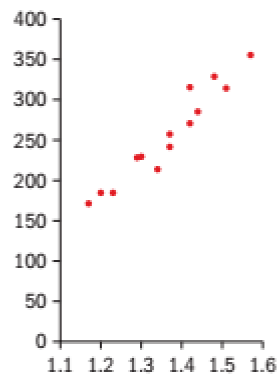
$y = a + bx$, where $b = \dfrac{S_{xy}}{S_{xx}} = \dfrac{-\frac{1679793545}{2}}{\frac{11491994663}{2}} = -0.1462$ and

$a = \bar{y} - b\bar{x} = \dfrac{256370}{8} + 0.1462 \times \dfrac{344550}{8} = 38342.9$ so $y = 38342.9 - 0.1462x$

$y = 38342.9 - 0.1462 \times 50000 = \$31032.90$

The make of car or price when new would be important information

4 a



b

x - Height (m)	y - Weight (kg)	x^2	y^2	xy
1.48	329	2.1904	108241	486.92

1.51	314	2.2801	98596	474.14
1.23	185	1.5129	34225	227.55
1.57	356	2.4649	126736	558.92
1.29	228	1.6641	51984	294.12
1.30	230	1.69	52900	299
1.37	257	1.8769	66049	352.09
1.17	171	1.3689	29241	200.07
1.2	185	1.44	34225	222
1.34	214	1.7956	45796	286.76
1.42	315	2.0164	99225	447.3
1.42	271	2.0164	73441	384.82
1.37	242	1.8769	58564	331.54
1.44	285	2.0736	81225	410.4
$\Sigma x = 19.11$	$\Sigma y = 3582$	$\Sigma x^2 = 26.2671$	$\Sigma y^2 = 960448$	$\Sigma xy = 4975.63$

$$S_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n}$$

$$S_{xx} = 26.2671 - \frac{19.11^2}{14} = 0.18195$$

$$S_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n}$$

$$S_{yy} = 960448 - \frac{3582^2}{14} = 43967.7$$

$$S_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}$$

$$S_{xy} = 4975.63 - \frac{19.11 \times 3582}{14} = 86.2$$

$y = a + bx$, where $b = \frac{S_{xy}}{S_{xx}} = \frac{86.2}{0.18195} = 473.757$ and

$a = \bar{y} - b\bar{x} = \frac{3582}{14} - 473.757 \times \frac{19.11}{14} = -390.821$ so $y = 473.757x - 390.821$

c $$r = \frac{S_{xy}}{\sqrt{(S_{xx}S_{yy})}}$$

$$r = \frac{86.2}{\sqrt{0.18195 \times 43967.7}} = 0.96375$$

there is a strong positive correlation

d $y = 473.757 \times 1.38 - 390.821 = 262.964$

Chapter review

1 Pick 15 students at random from each MYP class and 15 students at random from the DP group

2 **a** Pick 12 students at random from the whole medical school

b Pick 1.71 students from each year group at random

c Pick 2.4 students at random from year one and 1.6 students at random from each of the other year groups

d Ask for volunteers and pick the first 12

3 **i** Make sure the questions are clear

ii Make sure the questions are not leading

iii Ensure that the possible answers are applicable to everybody and no options are missed

4 **a** Number of pages

b Height of page

5 **a** Qualitative, continuous

b Quantitative, continuous

c Quantitative, discrete

d Quantitative, continuous

6

7 **a** Mean

$$=\frac{(5.5\times13)+(15.5\times16)+(25.5\times146)+(35.5\times139)+(45.5\times84)+(55.5\times32)+(65.5\times20)}{13+16+146+139+84+32+20}$$

$$=\frac{15885}{450}=35.3$$

Median group is 31-40

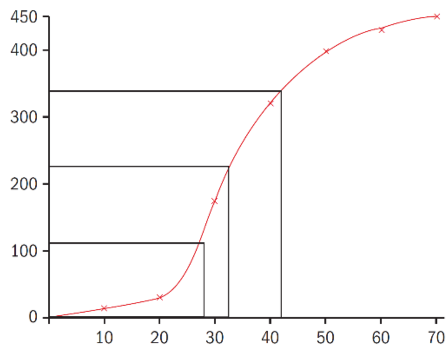
Modal group is 21-30

b $\frac{84+32+20}{450}\times100=30.2\%$

c

Number of words per sentence	Number of sentences	Cumulative frequency
1-10	13	13
11-20	16	29
21-30	146	175
31-40	139	314
41-50	84	398

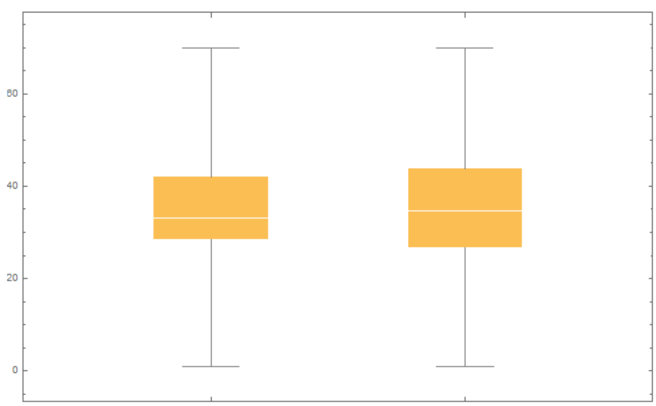
51-60	32	430
61-70	20	450



d

$$\begin{cases} \text{min} = 1 \\ Q_1 = 28.5 \text{ (approx)} \\ Q_2 = 33 \\ Q_3 = 42 \\ \text{max} = 70 \end{cases}$$

e



The diagram on the left is using the grouped data, the one on the right is using the five point summary from the graph

f The number of sentences is uniformly distributed within the interval.

g The box-and-whisker diagram using the grouped data is more skewed then the diagram using data from the graph.

8 a

Number of siblings	Frequency
0	14
1	28
2	11
3	5
4	0

5	2

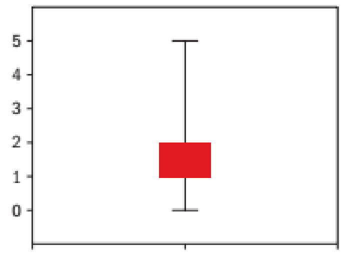
The data is left skewed.

b Mean $= \frac{0\times14+1\times28+2\times11+3\times5+5\times2}{60} = \frac{75}{60} = \frac{5}{4} = 1.25$

standard deviation

$= \sqrt{\frac{0^2\times14+1^2\times28+2^2\times11+3^2\times5+5^2\times2}{60} - \left(\frac{5}{4}\right)^2} = \sqrt{\frac{167}{60} - \frac{25}{16}} = 1.105 = 1.11$

c

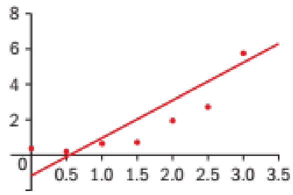


d Notice that the mean of all students is equal to the mean of the original 60 plus the mean of the new 32. Need to find x when $\frac{5}{4} + x = 1.25x$, so $\frac{5}{4} = 0.25x \Rightarrow x = 5$

9 a

x	0	0.5	1	1.5	2	2.5	3	3.5
y	0.6	0.45	0.8	0.85	1.4	1.65	2.4	2.85
y^2	0.36	0.2025	0.64	0.7225	1.96	2.7225	5.76	8.1225

b



c $y^2 = 2.13357x - 1.1725 \Rightarrow y = \sqrt{2.13357x - 1.1725}$

10 $y - y_0 = -0.5(x - x_0)$,

$y - 8 = -0.5(x - 1)$

$y = 8.5 - 0.5x$

so $y = 8.5 - 0.5x = 8.5 - 0.5\times7 = 5$

Exam-style questions

11 a As the mode is 5 there must be at least another 5. (1 mark)

So we have 1, 3, 5, 5, 6 with another number to be placed in order (1 mark)

The median will be the average of the 3rd and 4th pieces of data. (1 mark)

For this to be 4.5 the missing piece of data must be a 4.

Thus $a = 5$, $b = 4$ (2 marks)

b $\bar{x} = \frac{1+3+4+5+5+6}{6} = \frac{24}{6} = 4$ (2 marks)

12 a $\frac{\sum x}{10} = 70 \Rightarrow \sum x = 700$ (1 mark)

Let Steve's mass be s. $\frac{\sum x + s}{11} = 72$ (1 mark)

$700 + s = 792$ (1 mark)

So $s = 92$ kg (1 mark)

b IQR = 10 (1 mark)

$76 + 1.5 \times IQR = 76 + 15 = 91$ (1 mark)

So Steve's mass of 92 is greater than $1.5 \times IQR$, so is an outlier. (1 mark)

13 a 200 (1 mark)

b 35 (1 mark)

c Using mid-points 5, 15, 25… as estimates for each interval, (1 mark)

i estimate for mean is 22.25 (2 marks)

ii estimate for standard deviation is 11.6 (3 s.f.). (2 marks)

d Median is approximately the 100th piece of data

which lies in the interval $20 < h \le 30$. (1 mark)

Will be 15 pieces of data into this interval

Estimate is $20 + \frac{15}{50} \times 10 = 23$ (2 marks)

14 a **i** 7.5 (1 mark)

ii 6.125 (2 marks)

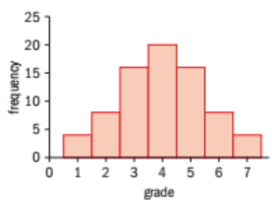
b **i** 6 (1 mark)

ii 6.9 (2 marks)

c Sally's had the greater median (1 mark)

d Rob's had the greater mean (1 mark)

15 a



(1 for scale, 1 for correctly drawn graph)

b **i** 4 **ii** 4 **iii** 4 (3 marks)

c The values of the median and the mean are the same due to the symmetry of the bar chart. (2 marks)

16 a $100 = 70m + c$

$140 = 100m + c$

$40 = 30m$ $\quad m = \frac{4}{3}$ $\quad c = \frac{20}{3}$ (3 marks)

b Positive (1 mark)

c Line goes through $(\bar{x}, \bar{y})$ (1 mark)

$\bar{y} = \frac{4}{3} \times 90 + \frac{20}{3} = \frac{380}{3}$ (2 marks)

d Estimate is $\frac{4}{3} \times 60 + \frac{20}{3} = \frac{260}{3}$ (2 marks)

17 a

x	13	14	15	16	16	17	18	18	19	19
y	2	0	3	1	4	1	1	2	1	2

(5 correct: 2 marks; all correct: 3 marks)

b $r = -0.0695(3sf)$ (2 marks)

c Very weak (negative) correlation so line of best fit is almost meaningless (1 mark)

It would be extrapolation to use this data to predict for a 25-year-old. (1 mark)

18 a **i** no change; $r = 0.87$ (1 mark)

ii no change; 15 (1 mark)

iii the scatter diagram has just been translated up by 5 and to the left by 4, so the PMCC and the gradient of y on x line of best fit are unchanged. (1 mark)

iv Strong, positive (2 marks)

b **i** no change; $r = 0.87$ (1 mark)

ii $2 \times 15 = 30$ (1 mark)

iii the scatter diagram has been stretched vertically by scale factor 2, so PMCC remains unchanged, but gradient of y on x line of best fit is doubled. (1 mark)

c **i** $r = -0.87$ (1 mark)

ii $\frac{15}{-3} = -5$ (1 mark)

iii the scatter diagram has been stretched horizontally by a factor of 3 and then reflected in the y-axis, so gradient becomes −5, but PMCC is unchanged. (2 marks)

iv Strong, negative (2 marks)

19 a **i** 0.849 (3sf) (2 marks)

ii strong, positive (2 marks)

iii $y = 0.937x + 0.242$ (2 marks)

b **i** 0.267 (3sf) (2 marks)

ii weak, positive (2 marks)

iii the Pearson product moment correlation coefficient is too small to make the line of best fit particularly meaningful when making predictions. (1 mark)

20 a $r = 0.979$ (3sf) (2 marks)

b Strong, positive (2 marks)

c i $y = 1.23x - 21.3$ (2 marks)

ii $x = 0.776y + 20.8$ (2 marks)

d $1.23 \times 105 - 21.3 = 108$ (1 mark)

e $0.776 \times 95 + 20.8 = 95$ (1 mark)

f It is extrapolation (1 mark)

6 Relationships in space: geometry and trigonometry

Skills check

1 $25^2 = (2x)^2 + x^2$

$25^2 = 5x^2$

$x^2 = 125$

$x = \sqrt{125}$

Area = $\sqrt{125} \times 2\sqrt{125} = 250\text{ cm}^2$

2 $\angle ACB = \angle PAQ$

As AB and PQ are parallel, lines BP and AQ meet AB and PQ at the same angle.

Therefore $\angle ABP = \angle BPQ$ and $\angle BAQ = \angle PQA$

All three angles are identical therefore, triangles are similar.

Exercise 6A

1 a i $(3,0,0)$ **ii** $(3,4,0)$ **iii** $(3,0,2)$ **iv** $(3,4,2)$

b Midpoint of OF: $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right) = \left(\frac{0+3}{2}, \frac{0+4}{2}, \frac{0+2}{2}\right) = (1.5,2,1)$

c Distance of OF $d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} = \sqrt{(3-0)^2+(4-0)^2+(2-0)^2}$

$= \sqrt{9+16+4} = \sqrt{29} \approx 5.4$

2 a $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right) = \left(\frac{-4+5}{2}, \frac{4-1}{2}, \frac{3+3}{2}\right) = (0.5,1.5,3)$

b $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right) = \left(\frac{-4-2}{2}, \frac{4+2}{2}, \frac{5+9}{2}\right) = (-3,3,7)$

c $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right) = \left(\frac{5-4}{2}, \frac{2-3}{2}, \frac{-4-8}{2}\right) = (0.5,-0.5,-6)$

d $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right) = \left(\frac{-5.1+1.4}{2}, \frac{-2+1.7}{2}, \frac{9+11}{2}\right) = (-1.85,-0.15,10)$

3 a $d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} = \sqrt{(4-2)^2+(3-3)^2+(1-5)^2} = \sqrt{4+0+16} = \sqrt{20} \approx 4.47$

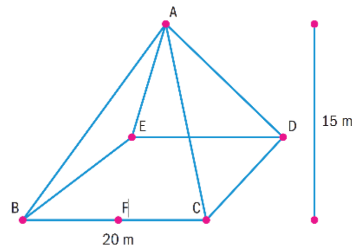
b $d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} = \sqrt{(2+3)^2+(4-7)^2+(-1-2)^2} = \sqrt{25+9+9} = \sqrt{43} \approx 6.56$

c $d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} = \sqrt{(1+1)^2+(-3-3)^2+(4+4)^2}$

$= \sqrt{4+36+64} = \sqrt{104} \approx 10.2$

d $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{(-2-2)^2 + (1+1)^2 + (3-3)^2} = \sqrt{16+4+0} = \sqrt{20} \approx 4.47$

4 a



b $\tan\theta = \dfrac{AO}{CO}$

$EC = \sqrt{DC^2 + ED^2} = \sqrt{20^2 + 20^2} = \sqrt{800} = 28.3$

Then

$CO = \dfrac{1}{2}EC = \dfrac{1}{2}28.3 = 14.14$

$\tan\theta = \dfrac{15}{14.14}$

$\theta = \tan^{-1}\dfrac{15}{14.14} = 46.7°$

c $\tan\theta = \dfrac{AO}{OM}$

$OM = \dfrac{1}{2}20 = 10$

$\tan\theta = \dfrac{15}{10}$

$\theta = \tan^{-1}\dfrac{15}{10} = 56.3°$

5 $BD = \sqrt{AB^2 + AD^2} = \sqrt{5^2 + 12^2} = \sqrt{25+144} = \sqrt{169} = 13$

$\tan\theta = \dfrac{FB}{BD} = \dfrac{4}{13}$

$\theta = \tan^{-1}\dfrac{4}{13} = 17.1°$

6 a $AC^2 = AB^2 + CB^2 = 4^2 + 4^2 = 32\text{cm}$

$AG = \sqrt{AC^2 + CG^2} = \sqrt{32+64} = 9.8\text{cm}$

b $\tan\theta = \dfrac{GC}{AC} = \dfrac{8}{\sqrt{32}}$

$\theta = \tan^{-1}\dfrac{8}{\sqrt{32}} = 54.7°$

c $\tan\theta = \frac{GE}{AE} = \frac{\sqrt{32}}{8}$

$\theta = \tan^{-1}\frac{\sqrt{32}}{8} = 35.3°$

$(90° - 54.7°)$

d $\sin\theta = \frac{4}{\sqrt{96}}$

$\theta = \sin^{-1}\frac{4}{\sqrt{96}} = 24.1°$

7 **a** $AC = \sqrt{8^2 + 8^2} = \sqrt{128} = 11.3\text{cm}$

b AM is the midpoint of AC, so

$AM = \frac{1}{2} \times AC = 5.7\text{cm}$

c $EA = \sqrt{EM^2 + AM^2} = \sqrt{64 + 32} = \sqrt{96} = 9.8\text{cm}$

d $\tan\theta = \frac{8}{\frac{1}{2}\sqrt{128}}$

$\theta = \tan^{-1}\frac{8}{\frac{1}{2}\sqrt{128}} = 54.7°$

Exercise 6B

Note that all answers should be provided to 3 significant figures

1 **a** $V = \frac{1}{3}A_{base}h = \frac{1}{3} \times 12 \times 12 \times 12 = 576\text{cm}^3$

$h_{face} = \sqrt{6^2 + 12^2} = 13.4$

$SA = A_{base} + 4A_{face} = (12 \times 12) + 4 \times \left(\frac{1}{2} \times 12 \times 13.4\right) = 466\text{cm}^2$

b $V = \frac{1}{3}A_{base}h = \frac{1}{3} \times 4 \times 5 \times 6 = 40\text{cm}^3$

$h_{face_1} = \sqrt{2^2 + 6^2} = 6.32 \; h_{face_2} = \sqrt{2.5^2 + 6^2} = 6.5$

$SA = A_{base} + 2A_{face_1} + 2A_{face_2} = (4 \times 5) + 2 \times \left(\frac{1}{2} \times 5 \times 6.32\right) + 2 \times \left(\frac{1}{2} \times 4 \times 6.5\right) = 77.6\text{cm}^2$

c $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times 3^2 \times 9 = 84.8\text{cm}^3$

$s = \sqrt{9^2 + 3^2} = 9.49$

$SA = \pi r^2 + \pi rs = \pi \times 3^2 + \pi \times 3 \times 9.49 = 118\text{cm}^2$

d $V = \frac{1}{3}\pi r^2 = \frac{1}{3}\pi \times 1 \times 3 = \pi = 3.14\text{cm}^3$

$s = \sqrt{1^2 + 3^2} = 3.16$

$SA = \pi r^2 + \pi rs = \pi \times 1 + \pi \times 1 \times 3.16 = 13.1\text{cm}^2$

e $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \times \left(\frac{16.4}{2}\right)^3 = 2310\text{cm}^3$

$SA = 4\pi r^2 = 4\pi\left(\frac{16.4}{2}\right)^2 = 845\text{cm}^2$

f $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \times \left(\frac{6}{2}\right)^3 = 113\text{cm}^3$

$SA = 4\pi r^2 = 4\pi(3)^2 = 113\text{cm}^2$

2 $V = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = \frac{1}{2}\left(\frac{4}{3}\pi\left(\frac{5.6}{2}\right)^3\right) = 46\text{cm}^3$

3 $V_{cyl} + V_{cone} = \pi r^2 h_{cyl} + \frac{1}{3}\pi r^3 h_{cone} = \pi \times 3.2^2 \times 9.1 + \frac{1}{3}\pi \times 3.2^2 \times 6.2 = 359\text{cm}^3$

4 $V_{clay} = V_{h1} - V_{h2} = \frac{1}{2} \times \frac{4}{3}\pi \times 9^3 - \frac{1}{2} \times \frac{4}{3}\pi \times 8^3 = 454\text{cm}^3$

5 $12\left(\frac{4\pi(3)^2}{3}\right) \div \pi(10)^2 = 1.44\text{ cm}$

6 $V_L = \frac{1}{3}\pi r_L^2 = \frac{1}{3}\pi \times \left(\frac{15}{2}\right)^2(15 + 45) = 3534.3\text{cm}^3$

$V_s = \frac{1}{3}\pi r_s^2 = \frac{1}{3}\pi \times \left(\frac{5}{2}\right)^2(15) = 98.17\text{cm}^3$

$V_f = V_L - V_s = 3436.1 = 3440\text{cm}^3$ (to 3 sf)

Exercise 6C

1 **a** $45° = \frac{45°\pi}{180°} = \frac{\pi}{4}$

b $60° = \frac{60°\pi}{180°} = \frac{\pi}{3}$

c $270° = \frac{270°\pi}{180°} = \frac{3\pi}{2}$

d $360° = 2\pi$

e $18° = \frac{18°\pi}{180°} = \frac{\pi}{10}$

f $225° = \frac{225°\pi}{180°} = \frac{5\pi}{4}$

g $80° = \frac{80°\pi}{180°} = \frac{4\pi}{9}$

h $200° = \frac{200°\pi}{180°} = \frac{10\pi}{9}$

i $120° = \frac{120°\pi}{180°} = \frac{2\pi}{3}$

j $135° = \frac{135°\pi}{180°} = \frac{3\pi}{4}$

2 a $\frac{\pi}{6}\frac{180°}{\pi} = 30°$

b $\frac{\pi}{10}\frac{180°}{\pi} = 18°$

c $\frac{5\pi}{6}\frac{180°}{\pi} = 150°$

d $3\pi\frac{180°}{\pi} = 540°$

e $\frac{7\pi}{20}\frac{180°}{\pi} = 63°$

f $\frac{4\pi}{5}\frac{180°}{\pi} = 144°$

g $\frac{7\pi}{4}\frac{180°}{\pi} = 315°$

h $\frac{14\pi}{9}\frac{180°}{\pi} = 280°$

i $\frac{5\pi}{3}\frac{180°}{\pi} = 300°$

j $\frac{13\pi}{4}\frac{180°}{\pi} = 585°$

3 a $10° = \frac{10°\pi}{180°} = 0.175$

b $40° = \frac{40°\pi}{180°} = 0.698$

c $25° = \frac{25°\pi}{180°} = 0.436$

d $300° = \frac{300°\pi}{180°} = 5.24$

e $110° = \frac{110°\pi}{180°} = 1.92$

f $75° = \frac{75°\pi}{180°} = 1.31$

g $85° = \frac{85°\pi}{180°} = 1.48$

h $12.8° = \frac{12.8°\pi}{180°} = 0.223$

i $37.5° = \frac{37.5°\pi}{180°} = 0.654$

j $1° = \frac{1°\pi}{180°} = 0.0175$

4 a $1 \times \frac{180°}{\pi} = 57.3°$

b $2 \times \frac{180°}{\pi} = 115°$

c $0.63 \times \frac{180°}{\pi} = 36.1°$

d $1.41 \times \frac{180°}{\pi} = 80.8°$

e $1.55 \times \frac{180°}{\pi} = 88.8°$

f $3 \times \frac{180°}{\pi} = 172°$

g $0.36 \times \frac{180°}{\pi} = 20.6°$

h $1.28 \times \frac{180°}{\pi} = 73.3°$

i $0.01 \times \frac{180°}{\pi} = 0.573°$

j $2.15 \times \frac{180°}{\pi} = 123°$

5 a $1 + 2A = \pi$

$A = \frac{\pi - 1}{2}$

b $V + 2 = \pi$

$V = \pi - 2$

Exercise 6D

1 a i $l = r\theta = 14 \times \frac{\pi}{2} = 7\pi\,\text{cm}$

ii $l = r\theta = 12 \times \frac{3\pi}{4} = 9\pi\,\text{m}$

iii $l = r\theta = 3 \times \frac{5\pi}{6} = \frac{5}{2}\pi\,\text{m}$

iv $l = r\theta = 15 \times \frac{14\pi}{9} = \frac{70}{3}\pi\,\text{cm}$

b i $A = \frac{1}{2}r^2\theta = \frac{1}{2} \times 14^2 \times \frac{\pi}{2} = 49\pi\,\text{cm}^2$

ii $A = \frac{1}{2}r^2\theta = \frac{1}{2} \times 12^2 \times \frac{3\pi}{4} = 54\pi\,\text{m}^2$

iii $A = \frac{1}{2}r^2\theta = \frac{1}{2} \times 3^2 \times \frac{5\pi}{6} = \frac{15}{4}\pi\,\text{m}^2$

iv $A = \frac{1}{2}r^2\theta = \frac{1}{2} \times 15^2 \times \frac{14\pi}{9} = 175\pi\,\text{cm}^2$

2 $A = \frac{1}{2}r^2\frac{\pi}{12} = 3\pi$

$r^2 = 3\pi \times 2 \times \frac{12}{\pi} = 72$

$r = \sqrt{72}\,\text{cm}$

3 a $A = \frac{1}{2}12^2\theta = 36\pi$

$\theta = \frac{36\pi}{12^2} \times 2 = \frac{\pi}{2}$

b $P = 2r + l$

$l = r\theta = 12 \times \frac{\pi}{2} = 6\pi$

$P = 2 \times 12 + 6\pi = (24 + 6\pi) = 42.8\text{m}$

4 $A_{shaded} = A_{sector} - A_{triangle}$

$A_{sector} = \frac{1}{2}r^2\theta = \frac{1}{2} \times 10^2 \times 1.5 = 75$

$A_{triangle} = \frac{1}{2}r^2\sin\theta = \frac{1}{2} \times 10^2 \sin 1.5 = 49.9$

$A_{shaded} = 75 - 49.9 = 25.1$

5 Arclength is travelled in one second, so we need $60l$ for total distance travelled.

$l = r\theta = 4 \times \frac{\pi}{12} = \frac{\pi}{3}$

Then $60l = 20\pi\text{m}$

6 $p = 2r + l = 2r + r\theta$

$r(2 + \theta) = p$

$r = \frac{p}{2 + \theta}$

7 $A_{quad} = \frac{1}{2}r_1^2\theta_1 = \frac{1}{2}\times 10^2 \times \frac{\pi}{2} = 78.54$

$A_{semi} = A_{quad} = \frac{1}{2}r_2^2\theta_2 = \frac{1}{2}r_2^2\pi = 78.54$

$r_2^2 = 78.54 \times \frac{2}{\pi} = 50$

$r_2 = \sqrt{50} = 7.07\text{cm}$

$p_{semi} = 2r + l = 2\times 7.07 + 7.07\times\pi = 36.35\text{cm}$

$r_1 - r_2 = 20 - 7.07 = 12.93\text{cm}$

Exercise 6E

1 **a** Quadrant II, $\theta = 180 - (-215 + 360) = 35°$

b Quadrant II, $\theta = \pi - \left(\frac{-5\pi}{4} + 2\pi\right) = \frac{\pi}{4}$

c Quadrant IV, $\theta = 4\pi - \frac{7\pi}{2} = \frac{\pi}{2}$

d Quadrant IV, $\theta = 2\pi - \frac{11\pi}{6} = \frac{\pi}{6}$

e Quadrant III, $\theta = 564° - 360° - 180° = 24°$

f Quadrant IV, $\theta = 22°$

g Quadrant II, $\theta = 3\pi - \frac{8\pi}{3} = \frac{\pi}{3}$

2

	$\sin\theta$	$\cos\theta$	$\tan\theta$	θ
A	0.5	–0.866	$\frac{0.5}{-0.866} = -0.577$	$\frac{5\pi}{6}$
B	$-\sin\frac{\pi}{6} = -0.5$	$\cos\frac{\pi}{6} = 0.866$	$-\frac{0.5}{0.866} = -0.577$	$\frac{11\pi}{6}$
C	0.3907	$\frac{0.3907}{-0.4245} = -0.9204$	–0.4245	2.74
D	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\frac{\pi}{4}$
E	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	–1	$\frac{3\pi}{4}$

3 **a** $\theta = \sin^{-1}0.6 = 36.9°, 143°$

b θ is undefined

c $\theta = \tan^{-1}(-2.36) = 293°, 113°$

4 $\cos\theta = \dfrac{3}{5}, \theta$ is in QIV

$\sin\theta = -\dfrac{4}{5}$ and $\tan\theta = -\dfrac{4}{3}$

Exercise 6F

1 $\theta = \cos^{-1} 0.45 = 1.104$ and $\theta = 2\pi - 1.104 = 5.18$

2 $\theta = \tan^{-1}(-0.56) = 5.77$ and $\theta = \pi - (2\pi - 5.77) = 2.63$

3 $\theta = \sin^{-1} 0.23 = 0.23$ and $\theta = \pi - 0.23 = 2.91$

4 theta is undefined

5 $\cos^2\theta + \cos\theta = 0$

$\cos\theta(\cos\theta + 1) = 0$

Then

$\cos\theta = 0$

gives that $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

and

$\cos\theta = -1$

gives that $\theta = \pi$

6 $2\cos^2\theta - 3\cos\theta + 1 = 0$

$(2\cos\theta - 1)(\cos\theta - 1) = 0$

Then

$\cos\theta = \dfrac{1}{2}$

so $\theta = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$ and

$\cos\theta = 1$

so $\theta = 0, 2\pi$

7 $4\sin^2\theta = 1$

$\sin\theta = \left(\dfrac{1}{4}\right)^{1/2} = \pm\dfrac{1}{2}$

then

$\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

8 $\frac{3\theta}{2} = \sin^{-1}(-0.62) = 5.61$

Note that $0 < \frac{3\theta}{2} < 3\pi$, so

$\frac{3\theta}{2} = 5.61, 3.81$

and so

$\theta = 3.74, 2.54$

9 $\tan 2\theta = -0.45555$

so $0 < 2\theta < 4\pi$ and

$\tan^{-1}(-0.45555) = 2\theta$

then

$2\theta = 2.714, 5.856, 8.997, 12.139\}$

So

$\theta = 1.36, 2.93, 4.50, 6.07$

where the first two angles are the angles for the negative tangent, and the last two are an added rotation to them $(+2\pi)$

Exercise 6G

1 a $p^2 = q^2 + r^2 - 2qr\cos P = 8^2 + 5^2 - 2\times 8\times 5\cos 30° = 19.72$

so

$p = \sqrt{19.72} = 4.44$

$\cos Q = \frac{r^2 + p^2 - q^2}{2rp} = \frac{5^2 + 4.44^2 - 8^2}{2\times 5\times 4.44} = -0.434$

$Q = \cos^{-1}(-0.434) = 116°$

and so

$R = 180° - 116° - 30° = 34°$

b $y^2 = x^2 + z^2 - 2xz\cos Y = 4^2 + 5^2 - 2\times 4\times 5\cos 95° = 44.49$

$y = \sqrt{44.49} = 6.67$

and so

$\cos X = \frac{y^2 + z^2 - x^2}{2yz} = \frac{6.67^2 + 5^2 - 4^2}{2\times 6.67\times 5} = 0.802$

$X = \cos^{-1} 0.802 = 36.7°$

and

$Z = 180° - 36.7° - 95° = 48.3°$

c $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{8^2 + 4^2 - 5^2}{2 \times 8 \times 4} = 0.8594$

so $A = 30.8°$

$\cos b = \dfrac{a^2 + c^2 - b^2}{2ac} = \dfrac{5^2 + 4^2 - 8^2}{2 \times 5 \times 4} = -0.575$

so $B = 125°$

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab} = \dfrac{5^2 + 8^2 - 4^2}{2 \times 8 \times 5} = 0.9125$

so $C = 24.2°$

2 Largest angle is opposite the longest side

$\cos\theta = \dfrac{3.9^2 + 2.3^2 - 4.5^2}{2 \times 2.3 \times 3.9}$

$\theta = \cos^{-1}\left(\dfrac{3.9^2 + 2.3^2 - 4.5^2}{2 \times 2.3 \times 3.9}\right) = 89.2°$

3 We have that

$3PQ = 2QR = 4RP$

so the smallest side is RP. Then

$$\cos PQR = \frac{PQ^2 + QR^2 - PR^2}{2PQQR} = \frac{\left(\frac{4}{3}RP\right)^2 + (2RP)^2 - RP^2}{2 \times \frac{4}{3} \times 2 \times RP^2} = \frac{43}{48}$$

$\cos^{-1}\dfrac{43}{48} = 26.4°$

4 $\cos 60° = \dfrac{5^2 + x^2 - (2x-1)^2}{2 \times 5x}$

or equivalently

$2 \times 5x \times \dfrac{1}{2} = 25 + x^2 - 4x^2 + 4x - 1$

or equivalently

$-3x^2 - x + 24 = 0$

which has solutions $x = -3, \dfrac{8}{3}$. We take the positive value as it is a distance. So

$x = \dfrac{8}{3}$

and $b = \dfrac{8}{3}$, $a = 2 \times \dfrac{8}{3} - 1 = \dfrac{13}{3}$.

Then we calculate the remaining angles as

$$\cos ABC = \frac{5^2 + \left(\frac{13}{3}\right)^2 - \frac{8}{3}}{2 \times 5 \times \frac{13}{3}} = \frac{37}{39}$$

so

$$ABC = \cos^{-1}\frac{37}{39} = 18.4°$$

and so

$$BCA = 180° - 18.4° - 60° = 102° \quad \text{(to 3sf)}$$

5 Note that $DAB = BCD$ and $CDA = CBA$. Then

$$2DAB + 2CDA = 360°$$

$$CDA = 180° - DAB$$

We use the cosine rule to get the relationships

$$\cos DAB = \frac{b^2 + a^2 - q^2}{2ba}$$

and

$$\cos CDA = \frac{a^2 + b^2 - p^2}{2ab}$$

Note that we also have that $\cos CDA = \cos(180° - DAB) = -\cos DAB$, so

$$\frac{b^2 + a^2 - q^2}{2ba} = \frac{-a^2 - b^2 + p^2}{2ab}$$

which rearranges to

$$p^2 + q^2 = 2(b^2 + a^2)$$

Exercise 6H

1 a $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

$$\frac{\sin 30°}{10} = \frac{\sin 125°}{b}$$

$$b = \frac{\sin 125°}{\sin 30°}10 = 16.4\text{ cm}$$

Then $C = 180° - 125° - 30° = 25°$

$$\frac{\sin 30°}{10} = \frac{\sin 25°}{c}$$

$$c = \frac{\sin 25°}{\sin 30°}10 = 8.45\text{ cm}$$

b $Q = 180° - 45° - 40° = 95°$

$$\frac{\sin 45°}{7} = \frac{\sin 40°}{p}$$

$$p = \frac{\sin 40°}{\sin 45°}7 = 6.36 \text{ cm}$$

and

$$\frac{\sin 45°}{7} = \frac{\sin 95°}{q}$$

$$q = \frac{\sin 95°}{\sin 45°}7 = 9.86 \text{ cm}$$

c $\dfrac{\sin 40°}{9} = \dfrac{\sin A}{7}$

$$\sin A = 7\frac{\sin 40°}{9} = 0.4999$$

$$A = \sin^{-1} 0.49999 = 30°$$

Then

$$B = 180° - 40° - 30° = 110°$$

and

$$\frac{\sin 40°}{9} = \frac{\sin 110°}{b}$$

and

$$b = 9\frac{\sin 110°}{\sin 40°} = 13.2 \text{ cm}$$

2 $\dfrac{\sin 15°}{150} = \dfrac{\sin Q}{80}$

$$\sin Q = 80\frac{\sin 15°}{150} = 0.138$$

$$Q = \sin^{-1} 0.138 = 7.93°$$

Then $R = 180° - 7.93° - 15° = 157.07°$

$$\frac{\sin 15°}{150} = \frac{\sin 157.07°}{r}$$

$$r = 150\frac{\sin 157.07°}{\sin 15°} = 225.8 \text{ km}$$

The error course took $80 + 150 = 230\text{km}$, which took

$$\frac{230}{400} = 0.575\text{h}$$

and the correct course would take

$$\frac{225.5}{400} = 0.5645\text{h}$$

Then the time difference is

$0.575 - 0.56375 = 0.0105\text{h}$

which in seconds is

$0.0105 \times 3600 = 38 \text{ seconds}$

3 The distance between the end of the lake and the balloon is given by

$$\tan 32^\circ = \frac{250}{x_1}$$

$x_1 = 400.1\text{m}$

The distance from the balloon to the beginning of the lake is

$$\tan(90^\circ - 68^\circ) = \frac{x_2}{250}$$

$x_2 = 250\tan 22^\circ = 101\text{m}$

Then the length of the lake is the difference between the two lengths,

$x_1 - x_2 = 400.1 - 101 = 299\text{m}$

4 $CBM = 180^\circ - 64^\circ = 116^\circ$

$BMC = 180^\circ - 116^\circ - 23^\circ = 41^\circ$

$$\frac{\sin 41^\circ}{15} = \frac{\sin 116^\circ}{MC}$$

$$MC = 15\frac{\sin 116^\circ}{\sin 41^\circ} = 20.5\text{m}$$

$$\frac{\sin 116^\circ}{20.5} = \frac{\sin 23^\circ}{MB}$$

$$MB = 20.5\frac{\sin 23^\circ}{\sin 116^\circ} = 8.91\text{m}$$

and

$$\frac{\sin 64^\circ}{MA} = \frac{\sin 90^\circ}{8.91}$$

$MA = 8.91\sin 64^\circ = 8\text{m}$

5 $$\frac{\sin 55^\circ}{27} = \frac{\sin ACB}{31}$$

$$\sin ACB = 31\frac{\sin 55^\circ}{27} = 0.9405$$

$ACB = \sin^{-1} 0.9405 = 70.1^\circ$

Then for this case, the third angle is $BAC = 180^\circ - 55^\circ - 70^\circ = 54.9^\circ$

Alternatively we can take the obtuse angle,

$ACB = 180^\circ - 70.1^\circ = 110^\circ$

and so the other triangle has angle

$BAC = 180° - 55° - 110° = 15°$

Exercise 6I

1 $A_{total} = A_{PQR} + A_{PRS}$

$A_{PQR} = \frac{1}{2} \times 10 \times 13 \times \sin 125° = 53.24$

$PR^2 = 10^2 + 13^2 - 2 \times 10 \times 13 \times \cos 125° = 418.13$

$PR = 20.45$

Then

$A_{RPS} = \frac{1}{2} \times 20.45 \times 15 \times \sin 70° = 144.13$

Then

$A_{total} = 53.24 + 144.13 = 197 \quad (3 \text{ sf})$

2 $\frac{\sin B}{104} = \frac{\sin 20°}{52}$

$B = \sin^{-1}\left(104 \frac{\sin 20°}{52}\right) = 43.16°$

In this case, the other angle is $C = 180° - 43.16° - 20° = 116.84°$, so

$A_1 = \frac{1}{2} \times 52 \times 104 \times \sin 116.84° = 2412.7 \text{cm}^2$

We can also take the obtuse angle for B, and so

$B = 180° - 43.16° = 136.84°$

Then the other angle is $C = 180° - 20° - 136.84° = 23.16°$, so

$A_2 = \frac{1}{2} \times 52 \times 104 \times \sin 23° = 1063.48 \text{cm}^2$

Therefore the differnce is $A_1 - A_2 = 2412.7 - 1063.48 = 1349.22 = 1350 \text{cm}^2 \quad (3 \text{ sf})$

3 $\cos YOZ = \frac{5^2 + 5^2 - 7^2}{2 \times 5 \times 5} = 0.02$

$YOZ = \cos^{-1} 0.02 = 88.9°$

$A_{YZO} = \frac{1}{2} \times 5 \times 5 \times \sin 88.9° = 12.5 \text{cm}^2$

$\cos XOY = \frac{5^2 + 5^2 - 3^2}{2 \times 5 \times 5} = 0.82$

$XOY = \cos^{-1} 0.82 = 34.9°$

$$A_{XOY} = \frac{1}{2} \times 5 \times 5 \times \sin 34.9^\circ = 7.15$$

Then

$$A_{total} = A_{YZO} + A_{XOY} = 12.5 + 7.15 = 19.7\text{cm}^2$$

4 $\tan 60^\circ = \frac{12}{CA}$.

$$CA = \frac{12}{\tan 60^\circ} = 6.93$$

$$\tan 55^\circ = \frac{12}{DA}$$

$$DA = \frac{12}{\tan 55^\circ} = 8.4$$

$$\cos CAD = \frac{CA^2 + DA^2 - CD^2}{2 \times CA \times DA} = \frac{6.93^2 + 8.4^2 - 15^2}{2 \times 6.93 \times 8.4} = -0.914$$

$$CAD = \cos^{-1} - 0.914 = 156^\circ$$

$$A_{CAD} = \frac{1}{2} \times CA \times DA \times \sin CAD = \frac{1}{2} \times 6.93 \times 8.4 \times \sin 156^\circ = 11.8\text{m}^2$$

5 **a** $A_{POQ} = \frac{1}{2} \times r^2 \times \sin\left(\frac{\pi}{4} + \frac{\pi}{6} + \frac{\pi}{4}\right) = \frac{1}{2} r^2 \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{4} r^2$

and $A_{ROS} = \frac{1}{2} r^2 \sin\frac{\pi}{6} = \frac{1}{4} r^2$

b $A_{sectorPQ} = \frac{1}{2} r^2\theta = \frac{1}{2} r^2\left(\frac{\pi}{4} + \frac{\pi}{6} + \frac{\pi}{4}\right) = \frac{\pi r^2}{3}$

$$A_{minorPQ} = A_{sectorPQ} - A_{POQ} = \frac{\pi r^2}{3} - \frac{\sqrt{3}}{4} r^2 = r^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$$

c $A_{sectorRS} = \frac{1}{2} r^2\theta = \frac{1}{2} r^2\left(\frac{\pi}{6}\right) = \frac{\pi r^2}{12}$

$$A_{minorRS} = A_{sectorRS} - A_{ROS} = \frac{\pi r^2}{12} - \frac{1}{4} r^2 = r^2\left(\frac{\pi}{12} - \frac{1}{4}\right)$$

d Shaded area should not include minor of RS (otherwise it's just $A_{minorPQ}$)

$$A_{shaded} = A_{minorPQ} - A_{minorRS} = r^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right) - r^2\left(\frac{\pi}{12} - \frac{1}{4}\right) = \frac{r^2}{4}\left(\pi + 1 - \sqrt{3}\right)$$

6 $Yacht : 24 \times 5 = 120$ nautical miles

$Catamaran : 15 \times 5 = 75$ nautical miles

$$\theta = 139^\circ - 37^\circ = 102^\circ$$

$$d^2 = 75^2 + 120^2 - 2 \times 75 \times 120 \cos 102^\circ = 23767.4$$

$d = \sqrt{23767.4} = 154.167 \approx 154$ nautical miles (3 sf)

The bearing is given by the relationship

$360° - 143° = 217°$

and the angle complementary to the bearing from the yacht to the origin. Then we find the other angle of the triangle as

$$\frac{\sin 102°}{154} = \frac{\sin A}{75}$$

$A = 151.6°$

where we chose the obtuse angle. Then the complementary angle to the bearing we are looking for is

$\theta = 217° - 151.6° = 65.4°$

Finally, the angle we are searching for is

$\phi = 180° - 65.4° = 115°$

Exercise 6J

1 $\sec\frac{\pi}{6} - \cot\frac{\pi}{3} = \frac{1}{\cos\frac{\pi}{6}} - \frac{\cos\frac{\pi}{3}}{\sin\frac{\pi}{3}} = \frac{1}{\frac{\sqrt{3}}{2}} - \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{3}$

2 $\csc\left(-\frac{2\pi}{3}\right) + 2\tan\frac{7\pi}{6} = \frac{1}{\sin\left(-\frac{2\pi}{3}\right)} + 2\frac{\sin\frac{7\pi}{6}}{\cos\frac{7\pi}{6}} = -\frac{2\sqrt{3}}{3} + \frac{2\sqrt{3}}{3} = 0$

3 **a** $\cos\theta\tan\theta = \cos\theta\frac{\sin\theta}{\cos\theta} = \sin\theta$

b $\cot\theta\sec\theta = \frac{\cos\theta}{\sin\theta}\frac{1}{\cos\theta} = \frac{1}{\sin\theta} = \csc\theta$

c $\csc\theta\tan\theta = \frac{1}{\sin\theta}\frac{\sin\theta}{\cos\theta} = \frac{1}{\cos\theta} = \sec\theta$

d $\cos\theta\sec^2\theta\sin\theta = \cos\theta\frac{1}{\cos^2\theta}\sin\theta = \frac{\sin\theta}{\cos\theta} = \tan\theta$

e $\frac{\tan\theta\cot\theta}{\sin\theta} = \frac{\frac{\sin\theta}{\cos\theta}\frac{\cos\theta}{\sin\theta}}{\sin\theta} = \frac{1}{\sin\theta} = \csc\theta$

f $\tan\theta\csc\theta = \frac{\sin\theta}{\cos\theta}\frac{1}{\sin\theta} = \frac{1}{\cos\theta} = \sec\theta$

4 $\csc\theta = \frac{1}{\sin\theta} = \frac{13}{5}$

Then

$$\sin\theta = \frac{5}{13}$$

This comes from a triangle of sides 5, 13 and $c = \sqrt{13^2 - 5^2} = 12$, then

$$\cos\theta = \frac{-12}{13}$$

and

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = -\frac{\frac{12}{13}}{\frac{5}{13}} = -\frac{12}{5}$$

5 $\sec\theta = \dfrac{1}{\cos\theta} = -\dfrac{5}{4}$

$$\cos\theta = -\frac{4}{5}$$

Then the third side can be calculated as $c = \sqrt{25-16} = 3$, and so

with $\sin\theta = -\dfrac{3}{5}$ which is negative to satisfy $\pi < \theta < 2\pi$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{3}{5}}{-\frac{4}{5}} = \frac{3}{4}$$

Exercise 6K

1 **a** $\sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$

b $\dfrac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \dfrac{\sin\theta}{\cos\theta} = \tan\theta$

c $\dfrac{\sqrt{1-\cos^2\theta}}{\cos\theta} = \dfrac{\sin\theta}{\cos\theta} = \tan\theta$

d $\dfrac{\cos\theta}{1-\cos^2\theta} = \dfrac{\cos\theta}{\sin\theta} = \cot\theta$

e $\sqrt{1+t^2} = \sqrt{1+\tan^2\theta} = \sqrt{1+\sec^2\theta - 1} = \sec\theta$

f $\dfrac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \dfrac{\tan\theta}{\sec\theta} = \dfrac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}} = \sin\theta$

2 **a** $a^2\sin^2\theta - a^2 = a^2\left(\sin^2\theta - 1\right) = a^2\cos^2\theta$

b $b\cot\theta\sqrt{b^2 + b^2\cot^2\theta} = b\cot\theta\sqrt{b^2(1+\cot^2\theta)} = b^2\cot\theta\sqrt{\csc^2\theta} = b^2\cot\theta\csc\theta$

$$b^2\frac{\cos\theta}{\sin\theta}\frac{1}{\sin\theta} = b^2\frac{\cos\theta}{\sin^2\theta}$$

c $\dfrac{b}{\sqrt{b^2+y^2}}=\dfrac{b}{\sqrt{b^2+b^2\cot^2\theta}}=\dfrac{b}{b\sqrt{1+\cot^2\theta}}=\dfrac{1}{\csc\theta}=\sin\theta$

d $\dfrac{\sqrt{a^2\sec^2\theta-a^2}}{a\sec\theta}=\dfrac{a\sqrt{\sec^2\theta-1}}{a\sec\theta}=\dfrac{a\sqrt{\tan^2\theta}}{a\sec\theta}=\dfrac{\tan\theta}{\sec\theta}=\dfrac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}}=\sin\theta$

3 **a** $3-3\cos\theta=2\sin^2\theta$

$3-3\cos\theta=2(1-\cos^2\theta)$

$2\cos^2\theta-3\cos\theta+1=0$

$(2\cos\theta-1)(\cos\theta-1)=0$

Then

$\cos\theta=\dfrac{1}{2}$

so $\theta=\dfrac{\pi}{3},\dfrac{5\pi}{3}$ and

$\cos\theta=1$

so $\theta=0,2\pi$

b $\sec\theta=\dfrac{1}{\cos\theta}=2$

so

$\cos\theta=\dfrac{1}{2}$

then $\theta=\dfrac{\pi}{3},\dfrac{5\pi}{3}$

c $\cos^2\theta+\sin\theta+1=0$

$1-\sin^2\theta+\sin\theta+1=0$

$\sin^2\theta-\sin\theta-2=0$

$(\sin\theta-2)(\sin\theta+1)=0$

Then

$\sin\theta=2$

cannot be a solution as it is outside the range of the sine function. We also have

$\sin\theta=-1$

so $\theta=\dfrac{3\pi}{2}$

d $\sec^2\theta=1+\tan\theta$

$\tan^2\theta+1-1-\tan\theta=0$

$\tan^2\theta - \tan\theta = 0$

$\tan\theta(\tan\theta - 1) = 0$

Then $\tan\theta = 0$, then $\theta = 0, 2\pi$ and

$\tan\theta = 1$

so $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

e $3\tan^2\theta - 5\sec\theta + 1 = 0$

$3\left(\sec^2\theta - 1\right) - 5\sec\theta + 1 = 0$

$3\sec^2\theta - 5\sec\theta - 2 = 0$

$(\sec\theta - 2)(3\sec\theta + 1) = 0$

So we have

$\sec\theta = \frac{1}{\cos\theta} = 2$

$\cos\theta = \frac{1}{2}$

Then $= \frac{\pi}{3}, \frac{5\pi}{3}$. The second equation gives

$\sec\theta = \frac{1}{\cos\theta} = -\frac{1}{3}$

$\cos\theta = -3$

which is outside of the range of cosine, so it gives no solutions.

f $2\cot\theta\cos\theta + 7 = 7\csc\theta$

$2\frac{\cos^2\theta}{\sin\theta} + 7 = 7\frac{1}{\sin\theta}$

We multiply both sides by $\sin\theta$ and get

$$2\cos^2\theta + 7\sin\theta = 7$$
$$2\left(1 - \sin^2\theta\right) + 7\sin\theta = 7$$
$$-5 - 2\sin^2\theta + 7\sin\theta = 0$$

Which gives us

$\Rightarrow \sin\theta = 1$ or $\sin\theta = \frac{5}{2}$

$\Rightarrow \theta = \frac{\pi}{2}$

4 a $\csc^2\theta - 1 = \cot^2\theta$

b $\frac{1 + 2\sin\theta\cos\theta}{\sin\theta + \cos\theta} = \frac{\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta}{\sin\theta + \cos\theta} = \frac{(\cos\theta + \sin\theta)^2}{\sin\theta + \cos\theta} = \cos\theta + \sin\theta$

c $$\frac{2\tan\theta+\sec^2\theta}{(\sin\theta+\cos\theta)^2}=\frac{2\tan\theta+\tan^2\theta+1}{(\sin\theta+\cos\theta)^2}=\left(\frac{\tan\theta+1}{\sin\theta+\cos\theta}\right)^2=\left(\frac{\frac{\sin\theta}{\cos\theta}+1}{\sin\theta+\cos\theta}\right)^2$$

$$=\left(\frac{\frac{\sin\theta+\cos\theta}{\cos\theta}}{\frac{\sin\theta+\cos\theta}{1}}\right)^2=\left(\frac{1}{\cos\theta}\right)^2=\sec^2\theta$$

Exercise 6L

1 $\sin(A+(-B))=\sin A\cos(-B)+\sin(-B)\cos A=\sin A\cos B-\sin B\cos A$

2 **a** Using the construction on pg. 38

$$\cos(A+B)\equiv\frac{OT}{OR}\equiv\frac{OP-TP}{OR}\equiv\frac{OP-SQ}{OR}\equiv\frac{OP}{OR}-\frac{SQ}{OR}$$

$$\equiv\frac{OP}{OQ}\frac{OQ}{OR}-\frac{SQ}{RQ}\frac{RQ}{OR}\equiv\cos B\cos A-\sin B\sin A$$

b $\cos(A+(-B))=\cos A\cos(-B)-\sin A\sin(-B)=\cos A\cos B+\sin A\sin B$

3 **a** $$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}=\frac{\sin A\cos B+\sin B\cos A}{\cos A\cos B-\sin A\sin B}$$

$$=\frac{\frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}}{1-\frac{\sin A}{\cos A}\frac{\sin B}{\cos B}}=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

b $$\tan(A+(-B))=\frac{\tan A+\tan(-B)}{1-\tan A\tan(-B)}=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

4 **a** $\sin 75°=\sin(45°+30°)=\sin 45°\cos 30°+\sin 30°\cos 45°$

$$=\frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2}+\frac{1}{2}\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{4}\left(\sqrt{3}+1\right)$$

b $$\tan 105°=\tan(60°+45°)=\frac{\tan 60°+\tan 45°}{1-\tan 60°\tan 45°}=\frac{\frac{1}{\sqrt{3}}+1}{1-\frac{1}{\sqrt{3}}}=\sqrt{3}+2$$

$105°$ is obtuse, so we take the negative value

$\tan 105°=-\left(\sqrt{3}+2\right)$

c $\sin 33°\cos 3°-\cos 33°\sin 3°=\sin(33°-3°)=\sin 30°=\frac{1}{2}$

d $\cos 75°\cos 15°+\sin 75°\sin 15°=\cos(75°-15°)=\cos(60°)=\frac{1}{2}$

5 **a** Note that the missing side of the triangle with angle A is $c=\sqrt{5^2-3^2}=4$, so $\cos A=\frac{-4}{5}$ as A is obtuse, and the side of the triangle with angle B is $c=\sqrt{13^2-5^2}=12$, so $\cos B=\frac{12}{13}$ as B is acute. Then

$$\cos(A-B)=\cos A\cos B+\sin A\sin B=-\frac{4}{5}\times\frac{12}{13}+\frac{3}{5}\times\frac{5}{13}=-\frac{33}{65}$$

b Note that

$$\tan A=\frac{\sin A}{\cos A}=\frac{\frac{3}{5}}{-\frac{4}{5}}=-\frac{3}{4}$$

and $\tan B=\dfrac{\sin B}{\cos B}=\dfrac{\frac{5}{13}}{\frac{12}{13}}=\dfrac{5}{12}$

Then $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}=\dfrac{-\frac{3}{4}+\frac{5}{12}}{1+\frac{3}{4}\times\frac{5}{12}}=-\dfrac{16}{63}$

6 **a** $\cot(A+B)=\dfrac{\cos(A+B)}{\sin(A+B)}=\dfrac{\cos A\cos B-\sin A\sin B}{\sin A\cos B+\sin B\cos A}$

Divide by $\sin A\sin B$ and get

$$=\frac{\dfrac{\cos A\cos B-\sin A\sin B}{\sin A\sin B}}{\dfrac{\sin A\cos B+\sin B\cos A}{\sin A\sin B}}=\frac{\cot A\cot B-1}{\cot A+\cot B}$$

b $\dfrac{\sin(A+B)}{\cos A\cos B}=\dfrac{\sin A\cos B+\sin B\cos A}{\cos A\cos B}=\tan A+\tan B$

c $\sec A+\tan A=\dfrac{1}{\cos A}+\dfrac{\sin A}{\cos A}=\dfrac{1+\sin A}{\cos A}$

d $\tan A+\cot A=\dfrac{\sin A}{\cos A}+\dfrac{\cos A}{\sin A}=\dfrac{\sin^2 A+\cos^2 A}{\cos A\sin A}=\dfrac{1}{\cos A\sin A}=\sec A\csc A$

e $\sec^2\theta+\csc^2\theta=\dfrac{1}{\cos^2\theta}+\dfrac{1}{\sin^2\theta}=\dfrac{\sin^2\theta+\cos^2\theta}{\sin^2\theta\cos^2\theta}=\dfrac{1}{\sin^2\theta\cos^2\theta}=\sec^2\theta\csc^2\theta$

f $\dfrac{\csc\theta-\cot\theta}{1-\cos\theta}=\dfrac{\frac{1}{\sin\theta}-\frac{\cos\theta}{\sin\theta}}{1-\cos\theta}=\dfrac{\frac{1-\cos\theta}{\sin\theta}}{1-\cos\theta}=\dfrac{1}{\sin\theta}=\csc\theta$

g $\csc x-\sin x=\dfrac{1}{\sin x}-\sin x=\dfrac{1-\sin^2 x}{\sin x}=\dfrac{\cos^2 x}{\sin x}=\cos x\cot x$

h $1+\cos^4 x-\sin^4 x=1+\cos^4 x-\left(\sin^2 x\right)^2=1+\cos^4 x-\left(1-\cos^2 x\right)^2$

$1+\cos^4 x-\left(1-2\cos^2 x+\cos^4 x\right)=2\cos^2 x$

i $\sec\theta+\tan\theta=\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}=\frac{1+\sin\theta}{\cos\theta}$

We multiply numerator and denominator by $1-\sin\theta$ and get

$$=\frac{1-\sin^2\theta}{\cos\theta(1-\sin\theta)}=\frac{\cos^2\theta}{\cos\theta(1-\sin\theta)}=\frac{\cos\theta}{1-\sin\theta}$$

j $\frac{\sin A\tan A}{1-\cos A}=\frac{\sin A\frac{\sin A}{\cos A}}{1-\cos A}=\frac{\frac{\sin^2 A}{\cos A}}{1-\cos A}=\frac{\frac{1-\cos^2 A}{\cos A}}{1-\cos A}=\frac{\frac{(1-\cos A)(1+\cos A)}{\cos A}}{1-\cos A}=\frac{1+\cos A}{\cos A}=1+\sec A$

7 **a** $\tan\left(A+(B+C)\right)=\frac{\tan A+\tan(B+C)}{1-\tan A\tan(B+C)}=\frac{\tan A+\frac{\tan B+\tan C}{1-\tan B\tan C}}{1-\tan A\frac{\tan B+\tan C}{1-\tan B\tan C}}$

$$=\frac{\tan A-\tan A\tan B\tan C+\tan B+\tan C}{1-\tan B\tan C-\tan A\tan B-\tan A\tan C}$$

b Substituting into our formula above

$$\tan(A+B+C)=\frac{\frac{1}{2}-\frac{1}{2}\times\frac{1}{5}\times\frac{1}{8}+\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}-\frac{1}{2}\times\frac{1}{5}-\frac{1}{2}\times\frac{1}{8}}=1$$

Hence $A+B+C=\frac{\pi}{4}$

c If A, B and C form the angles of a triangle, then $\tan(A+B+C)=0$, so

$$\frac{\tan A-\tan A\tan B\tan C+\tan B+\tan C}{1-\tan B\tan C-\tan A\tan B-\tan A\tan C}=0$$

or equivalently

$$\tan A-\tan A\tan B\tan C+\tan B+\tan C=0$$

or equivalently

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C$$

Exercise 6M

1 **a** $\tan A+\cot A=\frac{\sin A}{\cos A}+\frac{\cos A}{\sin A}=\frac{\sin^2 A+\cos^2 A}{\cos A\sin A}=\frac{1}{\cos A\sin A}=\frac{2}{2\cos A\sin A}=\frac{2}{\sin 2A}=2\csc 2A$

b $\frac{\sin 2A+\cos 2A+1}{\sin 2A-\cos 2A+1}=\frac{2\sin A\cos A+2\cos^2 A-1+1}{2\sin A\cos A-1+2\sin^2 A+1}=\frac{2\cos A\,(\sin A+\cos A)}{2\sin A(\cos A+\sin A)}=\cot A$

c $\frac{\cos 3X-\sin 3X}{1-2\sin 2X}=\frac{\cos(2X+X)-\sin(2X+X)}{1-2\sin 2X}=$

$$=\frac{\cos 2X\cos X-\sin 2X\sin X-\sin 2X\cos X-\sin X\cos 2X}{1-4\sin X\cos X}$$

$$=\frac{\cos 2X(\cos X\sin X)-\sin 2X\left(\sin X+\cos X\right)}{1-4\sin X\cos X}$$

$$= \frac{(\cos^2 X - \sin^2 X)\left(\cos X - \sin X\right) - 2\sin X\cos X(\sin X + \cos X)}{1 - 4\sin X\cos X}$$

$$= \frac{(\cos X + \sin X)(\cos X - \sin X)(\cos X - \sin X) - 2\sin X\cos X(\sin X + \cos X)}{1 - 4\sin X\cos X}$$

$$= \frac{(\cos X - \sin X)^2\left(\sin X + \cos X\right) - 2\sin X\cos X(\sin X + \cos X)}{1 - 4\sin X\cos X}$$

$$= \frac{(\sin X + \cos X)\ (\cos^2 X - 2\sin X\cos X + \sin^2 X - 2\sin X\cos X)}{1 - 4\sin X\cos X}$$

$$= \frac{\left(\sin X + \cos X\right)\ (\cos^2 X - 4\sin X\cos X + \sin^2 X\right)}{1 - 4\sin X\cos X}$$

$$= \frac{(\sin X + \cos X)\ (1 - 4\sin X\cos X)}{1 - 4\sin X\cos X} = \sin X + \cos X$$

d $\cot x - \csc 2x = \dfrac{\cos x}{\sin x} - \dfrac{1}{\sin 2x} = \dfrac{\cos x}{\sin x} - \dfrac{1}{2\sin x\cos x} = \dfrac{2\cos^2 x - \cos^2 x - \sin^2 x}{2\sin x\cos x}$

$$= \frac{\cos^2 x - \sin^2 x}{2\sin x\cos x} = \frac{\cos 2x}{\sin 2x} = \cot 2x$$

2 $\dfrac{\sin 4A}{\sin A} = \dfrac{2\sin 2A\cos 2A}{\sin A} = \dfrac{4\sin A\cos A\cos 2A}{\sin A} = \dfrac{4\sin A\cos A\ (2\cos^2 A - 1)}{\sin A}$

$= 8\cos^3 A - 4\cos A$

3 $\sin 2A\sec 2A = \dfrac{\sin 2A}{\cos 2A} = \tan 2A = \dfrac{2\tan A}{1 - \tan^2 A} = \dfrac{2\left(\frac{3}{4}\right)}{1 - \left(\frac{3}{4}\right)^2} = \dfrac{24}{7}$

4 $\cos 3X = \cos\left(2X + X\right) = \cos 2X\cos X - \sin 2X\sin X$

$\left(2\cos^2 X - 1\right)\cos X - 2\sin X\cos X\sin X = 2\cos^3 X - \cos X - 2\sin^2 X\cos X$

$= 2\cos^3 X - \cos X - 2\left(1 - \cos^2 X\right)\cos X$

$= 4\cos^3 X - 3\cos X$

$\sin 3X = \sin\left(2X + X\right) = \sin 2X\cos X + \sin X\cos 2X$

$= 2\sin X\cos^2 X + \sin X\ (1 - 2\sin^2 X)$

$= 2\sin X\ (1 - \sin^2 X) + \sin X - 2\sin^3 X$

$= -4\sin^3 X + 3\sin X$

5 $\cos 4A = 2\cos^2 2A - 1 = 2\left(2\cos^2 A - 1\right)^2 - 1 = 8\cos^4 A - 8\cos^2 A + 1$

6 $\tan 4A = \dfrac{2\tan 2A}{1-\tan^2 2A} = \dfrac{2\left(\dfrac{2\tan A}{1-\tan^2 A}\right)}{1-\left(\dfrac{2\tan A}{1-\tan^2 A}\right)^2} = \dfrac{\dfrac{4\tan A}{1-\tan^2 A}}{\dfrac{(1-\tan^2 A)^2-4\tan^2 A}{(1-\tan^2 A)^2}}$

$\dfrac{4\tan A(1-\tan^2 A)}{(1-\tan^2 A)^2-4\tan^2 A}$

7 **a** We use the formula obtained in 4 to express $\sin 3x$

$-4\sin^3 x + 3\sin x = \sin^2 x$

or equivalently

$\sin x(4\sin^2 x + \sin x - 3) = 0$

Then $\sin x = 0$ gives $x = 0, 2\pi$ and

$(4\sin^2 x + \sin x - 3) = (\sin x + 1)(4\sin x - 3)$

then $\sin x = -1$ is given by $x = \dfrac{3\pi}{2}$ and $\sin x = \dfrac{3}{4}$ gives $x = 0.848, 2.29$

b $\cot 2x = 2 + \cot x$

Note that

$\cot 2x = \dfrac{1}{\tan 2x} = \dfrac{1-\tan^2 x}{2\tan x}$

Then the equation becomes

$\dfrac{1-\tan^2 x}{2\tan x} = 2 + \dfrac{1}{\tan x}$

which simplifies into

$1 - \tan^2 x - 4\tan x - 2 = 0$

or equivalently

$\tan^2 x + 4\tan x + 1 = 0$

We use the quadratic formula to get that

$\tan x = -2 \pm \sqrt{3}$

so $x = \dfrac{23\pi}{12}, \dfrac{11\pi}{12}, \dfrac{19\pi}{12}, \dfrac{7\pi}{12}$

c We use the formula obtained in 4 to express $\cos 3x$

$4\cos^3 x - 3\cos x - 3\cos x = 2\cos^2 x - 1 + 1$

which simplifies into

$4\cos^3 x - 2\cos^2 x - 6\cos x = 0$

or equivalently

$2\cos x\ (2\cos^2 x - \cos x - 3) = 0$

so $2\cos x = 0$ gives $x = \frac{\pi}{2}, \frac{3\pi}{2}$ and

$(2\cos^2 x - \cos x - 3) = (2\cos x - 3)(\cos x + 1) = 0$

which gives for $\cos x = \frac{3}{2}$ no real results as it is outside of the range of the cosine function.

Finally, $\cos x = -1$, gives $x = \pi$

Exercise 6N

1 Reasoning

$\csc\theta = 1$ when $\sin\theta = 1$, so at $\theta = \frac{\pi}{2} \pm 2n\pi, n \in \mathbb{Z}$

$\csc\theta = -1$ when $\sin\theta = -1$, so at $\theta = \frac{3\pi}{2} \pm 2n\pi, n \in \mathbb{Z}$

$\csc\theta$ is unidentified when $\sin\theta = 0$, so at $\theta = 0 \pm n\pi, n \in \mathbb{Z}$ we have vertical asymptotes

$\csc\theta \to \infty$ as $\sin\theta \to 0^+$

$\csc\theta \to -\infty$ as $\sin\theta \to 0^-$

2 Reasoning

$\cot\theta = 1$ when $\tan\theta = 1$, so at $\theta = \frac{\pi}{4} \pm n\pi, n \in \mathbb{Z}$

$\cot\theta = -1$ when $\tan\theta = -1$, so at $\theta = \frac{3\pi}{4} \pm n\pi, n \in \mathbb{Z}$

$\cot\theta$ is unidentified when $\tan\theta = 0$, so at $\theta = 0 \pm n\pi, n \in \mathbb{Z}$ we have vertical asymptotes

$\cot\theta \to \infty$ as $\tan\theta \to 0^+$

$\cot\theta \to -\infty$ as $\tan\theta \to 0^-$

3 **a** $f(x) = \sin x$

Then $g(x) = 3f(4x)$. We apply the following transformations to $f(x)$:

We apply a vertical stretch of scale factor 3 parallel to the y-axis

We stretch the function $y = 3\sin x$ by a scale factor $\frac{1}{4}$ parallel to the x-axis

The period of the new function is therefore $\frac{2\pi}{4} = \frac{\pi}{2}$

The amplitude of $g(x)$ is 3 and the period of $g(x)$ is $\frac{\pi}{2}$

b The graph will the same as for $\sin(x)$ with a vertical shift of $\frac{3\pi}{2}$

c Let $f(x) = \sin x$, and $g(x) = -2f(\pi x)$. We apply the following transformations to $f(x)$:

Then we reflect $f(x)$ with respect to the x axis, and apply a vertical stretch of scale factor 2 parallel to the y-axis.

We stretch the function $y=-2\sin x$ by a scale factor $1/\pi$ parallel to the x-axis

The period of the new function is therefore $\frac{2\pi}{\pi}=2$

The amplitude of $g(x)$ is 2 and the period is 2

d Let $f(x)=\sin x$, then $g(x)=2f\left(4\left(x+\frac{\pi}{4}\right)\right)-1$. We apply the following transformations to $f(x)$:

We apply a vertical stretch of scale factor 2 parallel to the y-axis.

We stretch the function by a scale factor $\frac{1}{4}$ parallel to the x-axis

We shift horizontally by $\frac{\pi}{4}$

We shift vertically downwards by 1

4 **a** Let $f(x)=\cos x$ and $g(x)=2f(x)+2$. We apply the following transformations to $f(x)$:

We apply a vertical stretch of scale factor 2 parallel to the y-axis.

We shift the function vertically upwards by 2.

b Let $f(x)=\cos x$ and $g(x)=f(3x)-1$. We apply the following transformations to $f(x)$:

We stretch the function by a factor of $\frac{1}{3}$ parallel to the x-axis

We shift vertically downwards by 1

The new period is $\frac{2\pi}{3}$

c Let $f(x)=\cos x$ and $g(x)=-2f(3x)$. We apply the following transformations to $f(x)$:

We reflect along the x-axis.

We stretch the function by a factor of 2 parallel to the y-axis

We stretch the function by a factor of $\frac{1}{3}$ parallel to the x-axis

The new period is $\frac{2\pi}{3}$ and the amplitude is 2

d Let $f(x)=\cos x$ and $g(x)=3f(2x)+3$. We apply the following transformations to $f(x)$:

We stretch the function by a factor of 3 parallel to the y-axis

We stretch the function by a factor of $\frac{1}{2}$ parallel to the x-axis

The new period is $\frac{2\pi}{2} = \pi$ and the amplitude is 3

We shift the function vertically upwards by 3

5 There is one solution in the interval $0 \le x \le \pi$ as there is only one intersection between $f(x)$ and $g(x)$ in that interval.

Exercise 6O

1 a The amplitude is 5

the vertical shift is $+1$

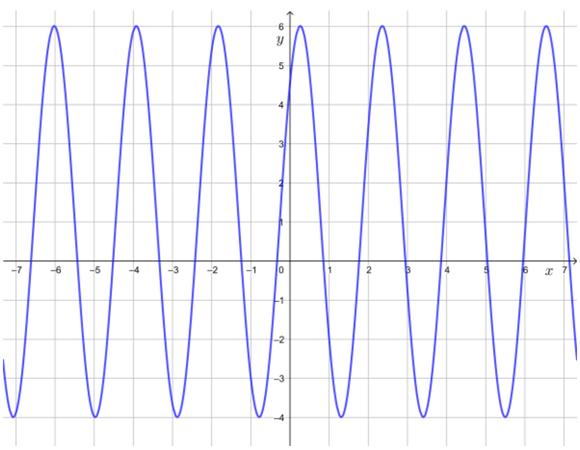
the horizontal/phase shift is $\frac{\pi}{12}$

the period is now $\frac{2\pi}{3}$

$5 = \frac{max - min}{2}$ and $1 = \frac{\text{max} + min}{2}$, then

the maximum value is 6, as

the minimum value is -4



b The amplitude is 2

the vertical shift is -2

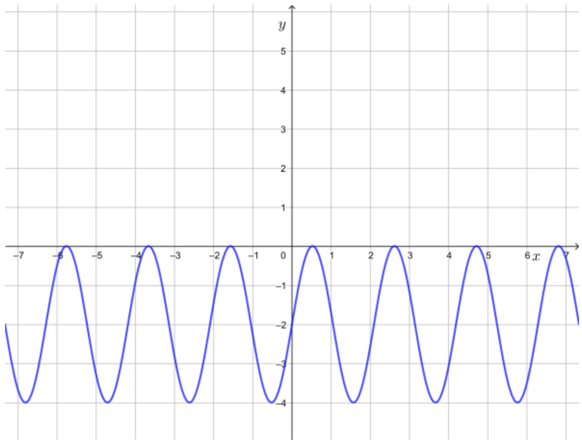
We rewrite the argument as $3\left(x + \frac{5\pi}{6}\right)$, so the horizontal/phase shift is $\frac{5\pi}{6}$

the period is now $\frac{2\pi}{3}$

$2 = \frac{max - min}{2}$ and $-2 = \frac{max + min}{2}$, then

the maximum value is 0, as

the minimum value is -4



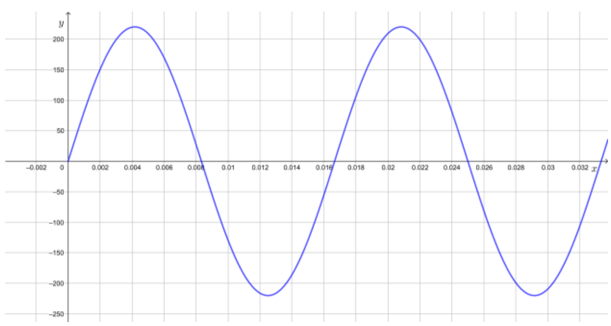
2 **a** The maximum value is 220, taken from the amplitude

b The minimum value is -220, as there is no vertical shift

c The amplitude of V is 220

d The period is given by $\frac{2\pi}{120\pi}=\frac{1}{60}$

e



3 **a** $a=\frac{\max-\min}{2}=\frac{14.4-1.2}{2}=6.6\text{m}$

vertical shift

$$d=\frac{\max+\min}{2}=\frac{14.4+1.2}{2}=7.8\text{m}$$

period of the function

$$\frac{2\pi}{b}=12\Rightarrow b=\frac{\pi}{6}$$

Then our function looks like

$$h(t)=6.6\sin\left(\frac{\pi}{6}(t+c)\right)+7.8$$

Note that $08:15$ is equivalent to $t=8.25$ hours. At $t=8.25$, $h=14.4$. Substituting into our equation for $h(t)$, gives us

$$6.6\sin\left(\frac{\pi}{6}(8.25+c)\right)+7.8=14.4$$

$$\sin\left(\frac{\pi}{6}(8.25+c)\right)=1$$

$$\left(\frac{\pi}{6}(8.25+c)\right)=\frac{\pi}{2}$$

$$8.25+c=3$$

$$c=-5.25$$

Then

$$h(t)=6.6\sin\left(\frac{\pi}{6}(t-5.25)\right)+7.8$$

b sketch of the graph, first minimum occurs at $t=2.25$

c The time intervals during which the boat could enter or leave the harbour on that particular day are calculated by plotting $y=5$ along with $h(t)$, and obtain the intervals for which $h(t)\geq 5$ over a period of 24 hours. This gives $0<t<0.0868$, $4.41<t<12.1$ and $16.4<t<24$.

4 **a** The minimum value is -3.5 and the maximum value is -2.5, this is a cosine function. We calculate the vertical shift and the amplitude as

$$d=\frac{-2.5-3.5}{2}=-3$$

$$a=\frac{-2.5+3.5}{2}=\frac{1}{2}$$

so

$$f(x)=\frac{1}{2}\cos x-3$$

b Horizontal shift, $\frac{\pi}{2}$, and we choose a cosine function. The amplitude is

$$a=\frac{7-3}{2}=2$$

and

$$d=\frac{7+3}{2}=5$$

is the vertical shift. Then the function is

$$f(x)=2\cos\left(x-\frac{\pi}{2}\right)+5$$

c We choose a cosine function. The amplitude is

$$a=\frac{2+4}{2}=3$$

and the vertical shift is

$d = \frac{2-4}{2} = -1$

The period is π, so $b = \frac{2\pi}{\pi} = 2$. Then the function is

$f(x) = 3\cos(2x) - 1$

d We choose a sine function, reflected along the x-axis. The amplitude is

$a = \frac{2+2}{2} = 2$

and there is no vertical shift. The period is $\frac{2\pi}{3}$, so $b = \frac{2\pi}{\frac{2\pi}{3}} = 3$. Then the function is

$f(x) = -2\sin 3x$

e This is a tangent function, shifted horizontally by $\frac{\pi}{4}$, so $f(x) = \tan\left(x - \frac{\pi}{4}\right)$

f This is a secant plot, shifted upwards by 1, where the asymptotes are at $x = -\pi, \pi$, etc… This corresponds to

$f(x) = \sec\frac{1}{2}x + 1$

Exercise 6P

1 **a** $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ **b** $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$ **c** $\theta = \pi$ **d** $\theta = \frac{\pi}{6}, \frac{7\pi}{6}$

e $\theta = 0, 2\pi$ **f** $\theta = \frac{\pi}{4}, \frac{7\pi}{4}$ **g** $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

2 $\cos^{-1} x = \frac{\pi}{2} - \frac{2\pi}{9} = \frac{5\pi}{18}$

3 **a** $\tan x < 0$ for $\frac{\pi}{2} < x < \pi$ and $\frac{3\pi}{2} < x < 2\pi$

b $\sec 2x < 0$ for $\frac{\pi}{4} < x < \frac{3\pi}{4}$ and $\frac{5\pi}{4} < x < \frac{7\pi}{4}$

c $\sin 4x > 3$ never happens as it is outside of the range of sine

4 **a** We solve

$y = \sin 2x$

for x, giving

$x = \frac{\sin^{-1} y}{2}$

Then

$f^{-1}(x)=\dfrac{\sin^{-1}x}{2}$

is defined for $x\in[0,1]$

In the case of $g(x)$,

$g^{-1}(x)=2x$.

and it is also well defined for all x

b $g^{-1}\left(\dfrac{1}{2}\right)=2\left(\dfrac{1}{2}\right)=1$

and

$f^{-1}g\left(\dfrac{\pi}{6}\right)=\dfrac{\sin^{-1}g\left(\dfrac{\pi}{6}\right)}{2}=\dfrac{\sin^{-1}\dfrac{\pi}{3}}{2}$

which has no real results

5 **a** $5=\dfrac{2\pi}{period}$

so the period is $\dfrac{2\pi}{5}$

b The amplitude is 6

c The sine function is symmetric about the origin

d We stretch the function by a factor of 6 parallel to the y-axis

We stretch the function by a factor of $\dfrac{1}{5}$ parallel to the x-axis

The new period is $\dfrac{2\pi}{5}$

There are no vertical shifts

Exercise 6Q

1 **a** For all of these, we graphically show the plot for the left hand side and the right hand side and find the points of intersection

$\csc^2\theta=3\cot\theta-1$

$\dfrac{1}{\sin^2\theta}=3\dfrac{\cos\theta}{\sin\theta}-1$

$1=3\cos\theta\sin\theta-\sin^2\theta$

$\cos^2\theta-3\cos\theta\sin\theta+2\sin^2\theta=0$

$(\cos\theta-2\sin\theta)(\cos\theta-\sin\theta)=0$

so we are searching for the solutions of

$\cos\theta = 2\sin\theta$

or equivalently

$2\tan\theta = 1$

$\tan\theta = \frac{1}{2}$

which has solutions $\theta = 0.464, 3.605$, and

$\cos\theta = \sin\theta$

which is true for $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

b $2\tan\theta = 3 + 5\cot\theta$

$2\tan\theta = 3 + \frac{5}{\tan\theta}$

$2\tan^2\theta = 3\tan\theta + 5$

$2\tan^2\theta - 3\tan\theta - 5 = 0$

$(2\tan\theta - 5)(\tan\theta + 1) = 0$

Then we get that

$\tan\theta = \frac{5}{2}$

which gives $\theta = 1.19, 4.33$ and

$\tan\theta = -1$

which gives $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

c $2\sec^2\theta - 3 + \tan\theta = 0$

$2\frac{1}{\cos^2\theta} - 3 + \frac{\sin\theta}{\cos\theta} = 0$

$2 - 3\cos^2\theta + \sin\theta\cos\theta = 0$

$2\sin^2\theta - \cos^2\theta + \sin\theta\cos\theta = 0$

We divide by $\cos^2\theta$ and get

$2\tan^2\theta + \tan\theta - 1 = 0$

or equivalently

$(\tan\theta + 1)\left(\tan\theta - \frac{1}{2}\right) = 0$

so $\tan\theta = -1$ gives $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$ and $\tan\theta = 1/2$ gives $\theta = 0.464, 3.61$

d $5\csc\theta + \cot\theta = 2\tan\theta$

$$\frac{5}{\sin\theta}+\frac{\cos\theta}{\sin\theta}=2\frac{\sin\theta}{\cos\theta}$$

$$5\cos\theta+\cos^2\theta=2\left(1-\cos^2\theta\right)$$

$$3\cos^2\theta+5\cos\theta-2=0$$

$$(\cos\theta+2)(3\cos\theta-1)=0$$

Then $\cos\theta=-2$ has no real solutions, and

$$\cos\theta=\frac{1}{3}$$

then $\theta=1.231, 5.052$

Exercise 6R

1 **a** $\frac{d}{dx}(\cos x)=\lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}$

$$=\lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}$$

$$=\lim_{h\to 0}\left(\cos x\frac{(\cos h-1)}{h}-\sin x\frac{\sin h}{h}\right)$$

$$=\cos x\lim_{h\to 0}\frac{(\cos h-1)}{h}-\sin x\lim_{h\to 0}\frac{\sin h}{h}$$

$$=0-\sin x(1)=-\sin x$$

b $\frac{d}{dx}(\sin 2x)=\lim_{h\to 0}\frac{\sin\left(2x+\frac{2h}{2}\right)-\sin(2x)}{\frac{h}{2}}$

$$=\lim_{h\to 0}\frac{\sin 2x\cos h+\sin h\cos 2x-\sin 2x}{\frac{h}{2}}$$

$$=\lim_{h\to 0}(\sin 2x\frac{\cos h-1}{\frac{h}{2}}+\cos 2x\frac{\sin h}{\frac{h}{2}})$$

$$=\sin 2x\lim_{h\to 0}\frac{\cos h-1}{\frac{h}{2}}+\cos 2x\lim_{h\to 0}\frac{\sin h}{\frac{h}{2}}$$

$$=\sin 2x\cdot 0+2\times\cos 2x=2\cos 2x$$

c $\frac{d}{dx}(\sin\frac{x}{3})=\lim_{h\to 0}\frac{\sin\left(\frac{x}{3}+\frac{3h}{3}\right)-\sin\left(\frac{x}{3}\right)}{3h}$

$$= \lim_{h\to 0}\frac{\sin\left(\frac{x}{3}\right)\cos h+\sin h\cos\left(\frac{x}{3}\right)-\sin\left(\frac{x}{3}\right)}{3h}$$

$$= \lim_{h\to 0}\left(\sin\left(\frac{x}{3}\right)\frac{\cos h-1}{3h}+\cos\left(\frac{x}{3}\right)\frac{\sin h}{3h}\right)$$

$$= \sin\left(\frac{x}{3}\right)\lim_{h\to 0}\frac{\cos h-1}{3h}+\cos\left(\frac{x}{3}\right)\lim_{h\to 0}\frac{\sin h}{3h}$$

$$= \sin\left(\frac{x}{3}\right)\cdot 0+\frac{1}{3}\cos\left(\frac{x}{3}\right)\cdot 1=\frac{1}{3}\cos\left(\frac{x}{3}\right)$$

d $$\frac{d}{dx}(\sin(2x+3)) = \lim_{h\to 0}\frac{\sin\left(2x+3+2\left(\frac{h}{2}-\frac{3}{2}\right)+3\right)-\sin(2x+3)}{\frac{h}{2}-\frac{3}{2}}$$

$$= \lim_{h\to 0}\frac{\sin(2x+3)\cos h+\sin h\cos(2x+3)-\sin(2x+3)}{\frac{h-3}{2}}$$

$$= \sin(2x+3)\lim_{h\to 0}\frac{\cos h-1}{\frac{h-3}{2}}+\cos(2x+3)\lim_{h\to 0}\frac{\sin h}{\frac{h-3}{2}}$$

$$= 0\cdot\sin(2x+3)+2\cos(2x+3)\cdot 1=2\cos(2x+3)$$

2 a $$\frac{d}{dx}(\tan x) = \lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h} = \lim_{h\to 0}\frac{\frac{\tan x+\tan h}{1-\tan x\tan h}-\tan x}{h}$$

$$= \lim_{h\to 0}\frac{\tan h\ (1+\tan^2 x)}{h} = \sec^2 x\lim_{h\to 0}\frac{\tan h}{h} = \sec^2 x\cdot 1 = \sec^2 x$$

b $$\frac{d}{dx}(\cot x) = \lim_{h\to 0}\frac{\cot(x+h)-\cot(x)}{h} = \lim_{h\to 0}\frac{\frac{\cos(x+h)}{\sin(x+h)}-\frac{\cos x}{\sin x}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{\sin x\cos(x+h)-\cos x\sin(x+h)}{\sin x\sin(x+h)}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{\sin(x-x-h)}{\sin x\sin(x+h)}}{h} = -1\lim_{h\to 0}\frac{1}{\sin x\sin(x+h)} = \frac{-1}{\sin x}\lim_{h\to 0}\frac{1}{\sin(x+h)} = \frac{-1}{\sin^2 x} = -\csc^2 x$$

c $$\frac{d}{dx}(\tan 3x) = \lim_{h\to 0}\frac{\tan\left(3x+\frac{h}{3}\right)-\tan 3x}{\frac{h}{3}}$$

$$= \lim_{h\to 0}\frac{\dfrac{\tan 3x + \tan\dfrac{h}{3}}{1-\tan 3x\tan\dfrac{h}{3}} - \tan 3x}{\dfrac{h}{3}} = \lim_{h\to 0}\frac{\tan\dfrac{h}{3}\,(1+\tan^2 3x)}{\dfrac{h}{3}}$$

$$= \sec^2 3x \lim_{h\to 0}\frac{\tan\dfrac{h}{3}}{\dfrac{h}{3}} = \sec^2 3x\cdot 3 = 3\sec^2 3x$$

Exercise 6S

1 a $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{\cos x}{\sin x}\right) = \dfrac{\dfrac{d}{dx}(\cos x)\cdot\sin x - \dfrac{d}{dx}(\sin x)\cdot\cos x}{\sin^2 x} = \dfrac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \dfrac{-1}{\sin^2 x} = -\csc^2 x$

b $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{1}{\sin x}\right) = -1\times(\sin x)^{-2}\times\cos x = \dfrac{-\cos x}{\sin^2 x} = -\cot x\csc x$

2 a $\dfrac{dy}{dx} = \dfrac{d}{dx}(\sin 2x) = \cos 2x\times 2 = 2\cos 2x$

b $\dfrac{dy}{dx} = \dfrac{d}{dx}(\cos(2x+1)) = -\sin(2x+1)\times 2 = -2\sin(2x+1)$

c $\dfrac{dy}{dx} = \dfrac{d}{dx}(\cos(8-3x)) = -\sin(8-3x)\times -3 = 3\cos(8-3x)$

d $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\cot\left(\dfrac{7-2x}{13}\right)\right) = \dfrac{d}{dx}\left(\dfrac{\cos\left(\dfrac{7-2x}{13}\right)}{\sin\left(\dfrac{7-2x}{13}\right)}\right)$

$$= \frac{-\sin\left(\dfrac{7-2x}{13}\right)\times\dfrac{-2}{13}\times\sin\left(\dfrac{7-2x}{13}\right) - \cos\left(\dfrac{7-2x}{13}\right)\times\dfrac{-2}{13}\times\cos\left(\dfrac{7-2x}{13}\right)}{\sin^2\left(\dfrac{7-2x}{13}\right)}$$

$$\frac{\dfrac{2}{13}\left(\sin^2\left(\dfrac{7-2x}{13}\right)+\cos^2\left(\dfrac{7-2x}{13}\right)\right)}{\sin^2\left(\dfrac{7-2x}{13}\right)} = \frac{2}{13}\csc^2\left(\frac{7-2x}{13}\right)$$

3 a $\dfrac{df(x)}{dx} = -\sin\left(x^5-3\right)\times 5x^4 = -5x^4\sin\left(x^5-3\right)$

b $\dfrac{df(x)}{dx} = \dfrac{d(\sin(x^2+1))^{-1}}{dx} = -1\times\sin^{-2}\left(x^2+1\right)\times\cos\left(x^2+1\right)\times 2x$

$$= \frac{-2x\cos\left(x^2+1\right)}{\sin^2\left(x^2+1\right)} = -2x\cot\left(x^2+1\right)\csc\left(x^2+1\right)$$

c $\dfrac{df(x)}{dx} = \dfrac{d}{dx}\left(\cos\left(4x^3-2x^2+7x+17\right)\right)^{-1}) =$

$$-1\times(\cos(4x^3-2x^2+7x+17))^{-2}\times-\sin(4x^3-2x^2+7x+17)\times(12x^2-4x+7)$$

$$=\frac{(12x^2-4x+7)\sin(4x^3-2x^2+7x+17)}{\cos^2(4x^3-2x^2+7x+17)}$$

$$=(12x^2-4x+7)\tan(4x^3-2x^2+7x+17)\sec(4x^3-2x^2+7x+17)$$

d $$\frac{df(x)}{dx}=\frac{d}{dx}\left(\frac{1}{\cos\left(\sqrt{e^x+1}\right)}\right)=$$

$$-1\times(\cos\left(\sqrt{e^x+1}\right))^{-2}\times-\sin\left(\sqrt{e^x+1}\right)\times\left(\frac{1}{2}\right)(e^x+1)^{-1/2}\times e^x$$

$$=\left(\frac{1}{2}\right)\frac{e^x\sin\left(\sqrt{e^x+1}\right)}{\sqrt{e^x+1}\,\cos^2\left(\sqrt{e^x+1}\right)}=\frac{1}{2\sqrt{e^x+1}}e^x\ \tan\sqrt{e^x+1}\sec\sqrt{e^x+1}$$

e $$\frac{df(x)}{dx}=\cos(\cos(\tan x))\times-\sin(\tan x)\times\sec^2 x=-\sec^2 x\sin(\tan x)\cos(\cos(\tan x))$$

Exercise 6T

1 **a** $y'=\cos x(2x+1)+2\sin x$

b $y'=-2(x+x^2)\sin 2x+(1+2x)\cos 2x$

c $$y'=\frac{-\sin x\times x-\cos x}{x^2}=\frac{-\cos x-x\sin x}{x^2}$$

d $$y'=\frac{2\times\sin 2x-2(2x+3)\cos 2x}{\sin^2 2x}=\frac{2\sin 2x-(4x+6)\cos 2x}{\sin^2 2x}$$

e $$y'=\frac{\sec^2 x\ (\sqrt{2-x})-\tan x\times\frac{1}{2}\times(2-x)^{-\frac{1}{2}}\times-1}{2-x}$$

$$=\frac{\sec^2 x}{\sqrt{2-x}}+\frac{1}{2}\times\frac{\tan x}{(2-x)^{3/2}}$$

2 **a** $y'=3\cos 3x$

Then evaluate at the point,

$$y'\left(\frac{\pi}{3}\right)=3\cos\left(\frac{3\pi}{3}\right)=-3$$

b $y'=-2\sin 2x$

Then evaluate at the point,

$$y'\left(\frac{5\pi}{4}\right) = -2\sin(2\frac{5\pi}{4}) = -2$$

c $y' = \sin x + (x-2)\cos x$

Then evaluate at the point

$$y'(0) = 0 - 2(1) = -2$$

d $y' = -3\cos x + 3x\sin x$

Then evaluate at the point

$$y'\left(\frac{\pi}{2}\right) = -3\cos\frac{\pi}{2} + \frac{3\pi}{2}\sin\frac{\pi}{2} = 0 + \frac{3\pi}{2} = \frac{3\pi}{2}$$

e $y' = 3x^2\tan x + x^3\sec^2 x$

Then evaluate at the point

$$y'\left(\frac{3\pi}{4}\right) = 3\left(\frac{3\pi}{4}\right)^2\tan\left(\frac{3\pi}{4}\right) + \left(\frac{3\pi}{4}\right)^3\sec^2\left(\frac{3\pi}{4}\right) = \frac{27\pi^3}{32} - \frac{27\pi^2}{16}$$

3 a $\sin^2\alpha + \cos^2\alpha = 1$, so the gradient is 0.

b $\dfrac{\tan\beta}{\sin\beta} = \sec\beta$, so the gradient is $\tan\beta\sec\beta$

c The gradient is $\dfrac{\sin^2 x + 2\sin^2 2x + 3\sin x\sin 2x + \cos^2 x + 2\cos^2 2x + 3\cos 2x\cos x}{(\cos x + \cos 2x)^2}$

$$= \frac{3(\cos x + 1)}{(\cos x + \cos 2x)^2}$$

Exercise 6U

1 a Let $f(x) = \cos x$, then $f^{-1}(x) = \cos^{-1} x = y$, then $f(y) = \cos y$ and so

$$\frac{dy}{dx} = \frac{1}{\frac{df}{dy}} = \frac{1}{-\sin y} = \frac{-1}{\sin(\cos^{-1} x)} = \frac{-1}{\sqrt{1-\cos^2((\cos^{-1} x))}} = \frac{-1}{\sqrt{1-x^2}}$$

Then, we use the chain rule

$$\frac{d}{dx}(\arccos 2x) = \frac{-1}{\sqrt{1-4x^2}} \times 2 = \frac{-2}{\sqrt{1-4x^2}}$$

b We use the form obtained in Investigation 8, and find

$$\frac{d}{dx}\left(\arcsin\frac{3}{2}x\right) = \frac{1}{\sqrt{1-\left(\frac{3}{2}x\right)^2}} \times \frac{3}{2} = \frac{3}{2\sqrt{1-\frac{9x^2}{4}}} = \frac{3}{\sqrt{4-9x^2}}$$

c Let $f(x) = \tan x$, then $f^{-1}(x) = \tan^{-1} x = y$, then $f(y) = \tan y$ and so

$$\frac{dy}{dx} = \frac{1}{\frac{df}{dy}} = \frac{1}{\sec^2 y} = \cos^2 y = \cos^2(\tan^{-1} x) = \frac{1}{1+x^2}$$

Then

$$\frac{d}{dx}(\tan^{-1}(2x+1)) = \frac{1}{1+(2x+1)^2} \times 2 = \frac{2}{4x^2+4x+2} = \frac{1}{2x^2+2x+1}$$

2 a $$\frac{dy}{dx} = \frac{d}{dx}(2x\arccos x) = 2\arccos x + 2x\left(\frac{-1}{\sqrt{1-x^2}}\right) = 2\arccos x - \frac{2x}{\sqrt{1-x^2}}$$

b $$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\arccos x}{2x}\right) = \frac{\frac{-1}{\sqrt{1-x^2}} \times 2x - 2\arccos x}{4x^2} = -\frac{\frac{2x}{\sqrt{1-x^2}} + 2\arccos x}{4x^2}$$

$$\frac{-1}{2x\sqrt{1-x^2}} - \frac{\arccos x}{2x^2}$$

c $$\frac{dy}{dx} = 2x\arctan 3x + 3 \times \frac{x^2-1}{1+9x^2}$$

3 a $$\frac{d}{dx}(\arcsin x + \arccos x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$$

This is valid because we are calculating two angles that add up to π in a right-angled triangle.

b $$\frac{d}{dx}(\arctan x + \arctan(-x)) = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$$

Here both inverse tangents correspond to the same angle, in different quadrants (due to the negative sign). So the rate of change between both is zero.

Exercise 6V

1 a $f(x) = \sin(3x+\pi)$

The tangent is

$$y = f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) + 0$$

We calculate

$$f'(x) = 3\cos(3x+\pi)$$

Then $f'\left(\frac{\pi}{3}\right) = 3\cos(\pi+\pi) = 3$

So the tangent equation is

$$y = 3\left(x - \frac{\pi}{3}\right) = 3x - \pi$$

The normal is

$$y = \frac{1}{f'\left(\frac{\pi}{3}\right)}\left(x - \frac{\pi}{3}\right) + 0$$

$$y = -\frac{1}{3}\left(x - \frac{\pi}{3}\right)$$

b $f(x) = \arccos 2x$

We calculate the derivative $f'(x) = -\frac{2}{\sqrt{1-4x^2}}$

Then $f'(0.05) = -2.01$

and $f(0.05) = 1.47$

The tangent equation is $y = -2.01(x - 0.05) + 1.47 = 1.57 – 2.01x$

and the normal is $y = -\frac{1}{-2.01}(x - 0.05) + 1.47 = 1.45 + 0.498x$

c $f(x) = x\sin 2x$

We calculate $f'(x) = \sin 2x + 2x\cos 2x$

and so $f'(-0.5) = -1.382$

and $f(0.5) = 0.421$

Then the tangent equation is $y = -1.382(x + 0.5) + 0.421 = -1.38x - 0.27$

and the normal is $y = -\frac{-1}{1.382}(x + 0.5) + 0.421 = 0.783 + 0.724x$

2 **a** $x\cos 2x = \tan(3x + \pi) - 1$

Point of intersection (using GDC) is $(0.298, 0.247)$.

b Normal to $y = x\cos 2x$

$f'(x) = \cos 2x - 2x\sin 2x$

so $f'(0.298) = 0.493$

Then the equation of the normal is $y = \frac{-1}{0.493}(x - 0.298) + 0.247 = 0.851 - 2.03x$

Normal to $y = \tan(3x + \pi) - 1$

$f'(x) = 3\sec^2(3x + \pi)$

so $f'(0.298) = 7.65$

Then the equation of the normal is $y = \frac{-1}{7.65}(x - 0.298) + 0.247 = 0.286 - 0.131x$

c From part **b**, the y-intercepts of each of the normal functions, which we call y_1 and y_2 respectively are

$y_1(0) = 0.851 - 2.03 \times 0 = 0.851$

and $y_2(0) = 0.286 - 0.131 \times 0 = 0.286$

The intersection between both functions is given by $x = 0.298$, as it is the point that they both share.

Then the two lines form a triangle, with sides

$c = 0.851 - 0.286 = 0.565$

$a = \sqrt{(0.298)^2 + (0.247 - 0.851)^2} = 0.674$

$b = \sqrt{(0.298)^2 + (0.247 - 0.286)^2} = 0.301$

Then $\cos C = \dfrac{0.674^2 + 0.301^2 - 0.565^2}{2(0.674)(0.301)} = 0.556$

$C = \cos^{-1} 0.556 = 1.073 = 56.2°$

3 a $y' = \dfrac{(\cos x - x\sin x)(x + \cos x) - x\cos x(1 - \sin x)}{(x + \cos x)^2} = \dfrac{\cos^2 x - x^2 \sin x}{(x + \cos x)^2}$

for $x \ge 0$

b Tangent at point $\left(\frac{\pi}{2}, 0\right)$. We calculate

$$y'\left(\frac{\pi}{2}\right) = \frac{\cos^2 \frac{\pi}{2} - \frac{\pi^2}{4} \sin \frac{\pi}{2}}{\left(\frac{\pi}{2} + \cos \frac{\pi}{2}\right)^2} = -1$$

Then the tangent is given by

$y = -1\left(x - \frac{\pi}{2}\right) + 0 = -\left(x - \frac{\pi}{2}\right)$

and the normal is given by

$y = -\frac{1}{-1}\left(x - \frac{\pi}{2}\right) + 0 = x - \frac{\pi}{2}$

4 a $y' = (8x)\arctan 2x + \dfrac{4x^2 + 1}{1 + x^2} 2$

b $y'(0.5) = 8(0.5)\arctan(2 \times 0.5) + 2 = \pi + 2$

and $y(0.5) = \left(4(0.5)^2 + 1\right)\arctan 0.5 = \frac{\pi}{2}$

and so the equation of the tangent at 0.5 is

$$y=(\pi+2)(x-0.5)+\frac{\pi}{2}=(2+\pi)x-1=5.14x-1$$

Exercise 6W

1 **a** $\frac{d\theta}{dt}=6\times\frac{1}{60}=0.1\,\text{rot}\,\text{s}^{-1}$

and $\tan\theta=\frac{y}{100}\Rightarrow y=100\tan\theta$

Then $\frac{dy}{dt}=100\sec^2\theta\frac{d\theta}{dt}=10\sec^2\theta$

We are measuring θ with respect to the shoreline, so when they are at right angles with respect to each other is exactly when $\theta=0$, so

$$\left.\frac{dy}{dt}\right|_{\theta=0}=10\,\text{m}\,\text{s}^{-1}$$

b $\tan\theta=\frac{50}{100}\Rightarrow\theta=\tan^{-1}\frac{1}{2}=0.464$

Then $\left.\frac{dy}{dt}\right|_{\theta=0.464}=12.5\,\text{m}\,\text{s}^{-1}$

c When the ray approaches being parallel to the light beam, the velocity of the light beam is increasing and is undetermined at the point where it is exactly parallel.

2 **a** $\frac{dy}{dt}=90\frac{km}{h}=90\frac{1}{60\text{min}}\times\frac{1\text{min}}{60\text{s}}\times\frac{1000\text{m}}{1\text{km}}=25\,\text{m}\,\text{s}^{-1}$

We measure the angle with respect to the horizontal of the camera 'line of sight'. Then we can write

$$\tan\theta=\frac{y}{30}\Rightarrow 30\tan\theta=y$$

Then $\frac{dy}{dt}=30\sec^2\theta\frac{d\theta}{dt}\Rightarrow\frac{25}{30}\cos^2\theta=\frac{d\theta}{dt}$

At $\theta=0$, the bird is directly in front of the camera, so we calculate

$$\frac{25}{30}\times1=0.833\text{ rot s}^{-1}$$

b We integrate $\frac{d\theta}{dt}$ function with respect to t to find

$$t^2=\frac{30}{25}\tan\theta$$

then at $t=0$ is when the bird is directly in front of the camera. So one second later is at $t=1$, gives $\theta=0.695$, so we evaluate $\frac{d\theta}{dt}$ at this value of θ and we get $0.491\,\text{rot}\,\text{s}^{-1}$

3 **a** We use the cosine rule to write a relationship between the decreasing angle θ and the decreasing side $6-y$ so we are modelling the decrease y as

$$\cos\theta=\frac{5^2+5^2-(6-y)^2}{2\times5\times5}=\frac{50-(6-y)^2}{50}$$

Then $(6-y)^2=50-50\cos\theta$

We differentiate both sides with respect to t and get

$$-2(6-y)\frac{dy}{dt}=50\sin\theta\frac{d\theta}{dt}$$

or equivalently, substituting with $\frac{dy}{dt}=0.1\,\text{cm}\,\text{s}^{-1}$ we get

$$(0.004y-0.024)\csc\theta=\frac{d\theta}{dt}$$

b In this case, we change the 6 for a 5 in the expression above, and write the rate of change as

$$\cos\theta=\frac{5^2+5^2-(5-y)^2}{2\times5\times5}=\frac{50-(5-y)^2}{50}=\frac{25+10y-y^2}{50}$$

$$50\cos\theta=25+10y-y^2$$

Then differentiating we get

$$-50\sin\theta\frac{d\theta}{dt}=10\frac{dy}{dt}-2y\frac{dy}{dt}=10(0.1)-2y(0.1)=1-0.2y$$

and so

$$\frac{d\theta}{dt}=\frac{0.2y-1}{50\sin\theta}=\frac{0.2y-1}{50}\csc\theta$$

4 **a** We have the relationship for the volume of the sphere

$$V=\frac{4}{3}\pi r^3$$

We differentiate with respect to t and get

$$\frac{dV}{dt}=\frac{4}{3}\pi 3r^2\frac{dr}{dt}$$

We use that $\frac{dV}{dt}=3\,\text{cm}^3\,\text{min}^{-1}$ and rearrange to get

$$\frac{3}{4\pi r^2}=\frac{dr}{dt}$$

so evaluating at $r=10$ gives

$$\frac{3}{4\pi(10)^2}=0.002387\,\text{cm}\,\text{min}^{-1}$$

b We have the relationship for the surface area of the sphere

$SA = 4\pi r^2$

and the relationship between the surface area and the volume as

$\frac{A}{3}r = V$.

so differentiating with respect to t gives the relationship

$$\frac{dA}{dt} \times \frac{1}{3}r + \frac{A}{3}\frac{dr}{dt} = \frac{dV}{dt}$$

For $r = 4.5,$ we use the relationship in a to get that

$$\frac{3}{4\pi r^2} = \frac{3}{4\pi (4.5)^2} = \frac{dr}{dt} = 0.0118$$

which we substitute into the rate of change for A, then

$$\frac{dA}{dt} \times \frac{1}{3} \times 4.5 + \frac{4\pi (4.5)^2}{3} 0.0118 = 3$$

Hence

$$\frac{dA}{dt} = 1.333\,\text{cm}^2\,\text{s}^{-1}$$

5 **a** take y to be the horizontal distance, and calculate the expression

$$\tan\theta = \frac{y}{10000}$$

Then

$$y = 10000\tan\theta$$

We convert $\frac{dy}{dt}$ into m s^{-1} to keep units consistent, so

$$\frac{dy}{dt} = 1025\frac{\text{km}}{\text{h}} = 284.72\ \text{m}\,\text{s}^{-1}$$

We differentiate the expression for y to get

$$\frac{dy}{dt} = 10000\sec^2\theta\frac{d\theta}{dt}$$

The angle we're looking at corresponds to

$$\theta = \tan^{-1}\frac{8000}{10000} = 38.7^\circ$$

Then we substitute into expression for the derivatives to get

$$\frac{d\theta}{dt} = \frac{284.72}{10000}\cos^2 38.7^\circ = 0.017\,\text{deg}\,\text{s}^{-1}$$

b When the plane is directly above the radar, $\theta = 0$, so

$$\frac{d\theta}{dt} = \frac{284.72}{10000} \times 1 = 0.028\,\text{deg}\,\text{s}^{-1}$$

6 **a** We can write the distance between the camera z and the train as

$$z = \sqrt{y^2 + 2^2} = \sqrt{y^2 + 4}$$

Then we use the chain rule to get

$$\frac{dz}{dt} = \frac{1}{2}\left(y^2 + 4\right)^{-\frac{1}{2}}(2y)\frac{dy}{dt} = \frac{y}{\sqrt{y^2 + 4}}\frac{dy}{dt}$$

We evaluate at $z = 4$ so $4 = \sqrt{y^2 + 4}$, so $y = 3.46$. Substituting into the above equation we get that

$$\frac{dz}{dt} = \frac{3.46}{4} \times 75 = 64.9\frac{\text{km}}{\text{h}}$$

b So we are looking for $\frac{d\theta}{dt}$ when $y = 3.46$, where $\tan\theta = \frac{y}{2}$

so $\theta = \tan^{-1}\frac{3.46}{2} = 60°$. We differentiate with respect to t and get

$$\sec^2\theta\frac{d\theta}{dt} = \frac{dy}{dt} \times \frac{1}{2}$$

which then substituting and converting 75 km/h to 20.8 m/s gives

$$\sec^2 60°\frac{d\theta}{dt} = 20.8 \times \frac{1}{2}$$

which gives

$$\frac{d\theta}{dt} = 2.6 \text{ degrees per second.}$$

7 We have that

$$\cos\theta = \frac{y}{3}$$

where y is the horizontal distance in metres. Then

$$y = 3\cos\theta$$

We convert to meters per second, $\frac{dy}{dt} = \frac{6}{100} = 0.06\,\text{m/s}$. We differentiate with respect to t and get

$$-3\sin\theta\frac{d\theta}{dt} = \frac{dy}{dt}$$

The angle corresponding to the horizontal distance of 1 m is

$$\cos\theta = \frac{1}{3}$$

so $\theta = \cos^{-1}\frac{1}{3} = 1.23$. We substitute all together is

$$-3\sin 1.23\,\frac{d\theta}{dt} = 0.06$$

then $\frac{d\theta}{dt} = -0.0212$

Chapter review

1 $V_{cube} = 4^3 = 64\text{cm}^3$

we equate to the volume of the sphere

$64 = \frac{4}{3}\pi r^3$

so

$r^3 = \frac{48}{\pi}$

then

$r = 2.48\text{cm}$

2 **a** $V_{tot} = V_{cone} + V_h$

$V_{cone} = \frac{1}{3}\pi r^2 h_{cone} = \frac{1}{3}\pi(6)^2(14-6) = 301.6\text{cm}^3$

$V_h = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = \frac{1}{2}\left(\frac{4}{3}\pi(6)^3\right) = 452.4\text{cm}^3$

$V_{tot} = 301.6 + 452.4 = 754\text{cm}^3$

b For the surface area we need to be careful to not count the circular base.

The cone slant height is

$s = \sqrt{8^2 + 6^2} = 10$

$SA_{cone} = \pi rs = \pi(6)(10) = 188.5\text{cm}^2$

$SA_h = \frac{1}{2}(4\pi r^2) = \frac{1}{2}\left(4\pi(6)^2\right) = 226.2\text{cm}^2$

Then

$SA_{tot} = 188.5 + 226.2 = 415\text{cm}^2$

3 **a** $\cos CAB = \frac{3^2 + 5^2 - 3^2}{2\times 3\times 5} = 0.833$

then $\cos^{-1} 0.833 = 0.586 = 33.6°$.

b **i** Note that $AY = XB$ as this is an isosceles triangle. Then

$AB - BY = AY$

$5 - 3 = 2$

Then

$AB = AY + XB + XY$

$5 = 2 + 2 + XY$

Then $XY = 1$.

So the length of the perimeter is

$CY + CX + XY = 4.52$

ii $A_{ACX} = \frac{1}{2}r^2\theta = \frac{1}{2}(3)^2(0.833) = 3.75\text{cm}^2$

iii Height of triangle ABC is

$h = \sqrt{9 - 2.5^2} = 1.66$

$A_{ABC} = \frac{5 \times 1.66}{2} = 4.15\text{cm}^2$

Then

$A_{ABC} - A_{ACX} = 4.15 - 3.749 = 0.401 = A_{CXB} = A_{ACY}$

$A_R = A_{ABC} - 2A_{CXB} = 3.35\text{cm}^2$

4 **a** $6\sin^2 x = 5 + \cos x$

$6(1 - \cos^2 x) = 5 + \cos x$

$6\cos^2 x + \cos x - 1 = 0$

$(3\cos x - 1)(2\cos x + 1) = 0$

and so $\cos x = \frac{1}{3}$ and $\cos x = -\frac{1}{2}$

b $x = \cos^{-1}\frac{1}{3} = 360° - 70.529 = 289.5°, 430.5°,$ and $x = \cos^{-1} - \frac{1}{2} = 180° + 60° = 240°, 480°$

5 $4\tan^2 x + 12\sec x + 1 = 0$

$4\frac{\sin^2 x}{\cos^2 x} + 12\frac{1}{\cos x} + 1 = 0$

$4\sin^2 x + 12\cos x + \cos^2 x = 0$

$4(1 - \cos^2 x) + 12\cos x + \cos^2 x = 0$

$3\cos^2 x - 12\cos x - 4 = 0$

Then

$\cos x = 4.31$

which is undefined, and

$\cos x = -0.309$

$\cos^{-1}(-0.309) = 108°$

and $-108°$

6 $\tan 3A = \tan(2A + A) = \dfrac{\tan A + \tan 2A}{1 - \tan A \tan 2A}$

$$= \frac{\tan A + \dfrac{2\tan A}{1-\tan^2 A}}{1 - \tan A \dfrac{2\tan A}{1-\tan^2 A}} = \frac{\tan A\ (3-\tan^2 A)}{1-3\tan^2 A}$$

7 We use a change of variables. Let $\phi = 3\theta$. Then we can rewrite the equation as

$$\cos\left(\frac{4}{3}\phi\right) = \cos\phi$$

or equivalently

$$\cos\left(2\pi - \frac{4}{3}\phi\right) = \cos\phi$$

$$n\pi + 2\pi = \left(\frac{3}{3} + \frac{4}{3}\right)\phi$$

Then

$$\theta = \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}, 0, 2\pi$$

If we substitute into the equation $\cos 4\theta = \cos 3\theta$, and apply the identities for $\cos(2\theta + 2\theta)$ and $\cos(2\theta + \theta)$ we get the equation

$$8\cos^3\theta + 4\cos^2\theta - 4\cos\theta - 1 = 0$$

so the roots to the equation are precisely $\cos\dfrac{2\pi}{7}, \cos\dfrac{4\pi}{7}, \cos\dfrac{6\pi}{7}$ where $x = \cos\theta$

Substitute the inverse roots to show the required equality.

8 **a** $\sin(x + 60°) = \cos x$

$$\sin x \cos 60° + \sin 60° \cos x = \cos x$$

$$\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = \cos x$$

$$\frac{1}{2}\tan x + \frac{\sqrt{3}}{2} = 1$$

$$\tan x = \left(1 - \frac{\sqrt{3}}{2}\right) \times 2$$

$$x = 15°, 195°$$

b $\tan(A - x) = \dfrac{\tan A - \tan x}{1 + \tan A \tan x} = \dfrac{2}{3}$

$$\frac{3 - \tan x}{1 + 3\tan x} = \frac{2}{3}$$

$$3(3 - \tan x) = 2(1 + 3\tan x)$$

$$9\tan x = 7$$

$$x = \tan^{-1}\left(-\frac{7}{9}\right) = 142°, 322°$$

9 $\sin 3A = \sin(A + 2A) = \sin A\cos 2A + \sin 2A\cos A$

$$= \sin A(1 - 2\sin^2 A) + 2\sin A\ (1 - \sin^2 A)$$

$$= \sin A - 2\sin^3 A + 2\sin A - 2\sin^3 A = -4\sin^3 A + 3\sin A$$

10 We can write the equation for the angles, relabelling $\arcsin x = \theta_1$ and $\arccos\frac{x}{2} = \theta_2$

$$\theta_1 + \theta_2 = \frac{5\pi}{6}$$

Then we calculate the sine on both sides (could be cosine or tangent, any function will do) and get

$$\sin(\theta_1 + \theta_2) = \sin\left(\frac{5\pi}{6}\right)$$

$$\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1 = \frac{1}{2}$$

We complete the triangle and get that

$$\frac{x}{1} \times \frac{x}{2} + \sqrt{1 - \frac{x^2}{4}}\sqrt{1 - x^2} = \frac{1}{2}$$

or equivalently

$$\left(1 - \frac{x^2}{4}\right)(1 - x^2) = \left(\frac{x^2}{2} - \frac{1}{2}\right)^2$$

which simplifies into

$$3x^3 - 3 = 0$$

which has solutions

$$x^2 = 1$$

and so

$$x = \pm 1$$

11 $\tan\frac{A}{2}\tan\frac{B-C}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} \times \frac{\sin\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B-C}{2}\right)}$

Note that

$$A + B + C = \pi$$

as ABC are the angles of a triangle. Then

$$\frac{A}{2} = \frac{\pi - (B + C)}{2}$$

so

$$\sin\frac{A}{2} = \sin\left(\frac{\pi}{2} - \left(\frac{B+C}{2}\right)\right) = \cos\left(\frac{B+C}{2}\right)$$

and

$$\cos\frac{A}{2} = \cos\left(\frac{\pi}{2} - \left(\frac{B+C}{2}\right)\right) = \sin\left(\frac{B+C}{2}\right)$$

Then we substitute back into the first equation to get

$$\frac{(\cos\frac{B}{2}\cos\frac{C}{2} - \sin\frac{B}{2}\sin\frac{C}{2})(\sin\frac{B}{2}\cos\frac{C}{2} - \sin\frac{C}{2}\cos\frac{B}{2})}{\left(\sin\frac{B}{2}\cos\frac{C}{2} + \sin\frac{C}{2}\cos\frac{B}{2}\right)\left(\cos\frac{B}{2}\cos\frac{C}{2} + \sin\frac{B}{2}\sin\frac{C}{2}\right)}$$

Note that

$$\sin\frac{B}{2}\cos\frac{B}{2} = \frac{1}{2}\sin B$$

and equivalently for C. We multiply the brackets and substitute with the form for $\sin B$ to get

$$\frac{\sin B(\cos^2\frac{C}{2} + \sin^2\frac{C}{2}) - \sin C(\cos^2\frac{B}{2} + \sin^2\frac{B}{2})}{\sin B(\cos^2\frac{C}{2} + \sin^2\frac{C}{2}) + \sin C(\cos^2\frac{B}{2} + \sin^2\frac{B}{2})} = \frac{\sin B - \sin C}{\sin B + \sin C}$$

Finally, we have that the sine rule holds, so we rewrite in terms of only $\sin C$ to get

$$\frac{\frac{b}{c}\times\sin C - \sin C}{\frac{b}{c}\sin C + \sin C} = \frac{\frac{b}{c} - 1}{\frac{b}{c} + 1} = \frac{b-c}{b+c}$$

12 The angle CAB is

$$CAB = 247^\circ - 36^\circ = 211^\circ$$

and the two sides adjacent to it are 120 and 234, so the distance to C corresponds to the third side, calculated with the cosine rule as

$$c^2 = 120^2 + 234^2 - 2\times 120\times 234\cos 211^\circ$$

$$c = 342.5\text{km}$$

The bearing of town B from C is given by the complementary angle to

$$360^\circ - 36^\circ - 211^\circ = 113^\circ$$

so the angle we are searching for is

$$180^\circ - 113^\circ = 67^\circ$$

13 a The acceleration is given by the double derivative of x with respect to t, so (assuming a constant)

$$\frac{dx}{dt} = -2a\cos t\sin t = -a\sin 2t$$

so

$$\frac{d^2x}{dt^2} = -2a\cos 2t$$

b The particle is at rest where the velocity is zero, so where

$$\frac{dx}{dt} = 0 = -a\sin 2t$$

Then we have that $2t = n\pi$, and so $t = \frac{n\pi}{2}$

c First we check where the acceleration is zero, which will give us the turning points of the velocity, as

$$\frac{d^2x}{dt^2} = -2a\cos 2t = 0$$

so $2t = \frac{\pi}{2} + n\pi$ and so $t = \frac{\pi}{4} + \frac{n\pi}{2}$. To find whether it is a maximum or a minimum, we check that the velocity as time approaches $t = \frac{\pi}{4}$ it decreases from both the right and the left, so this is a minimum. For $t = \frac{3\pi}{4}$. we have the opposite behaviour, so this is a maximum. This is for positive a. If we have a negative a, the maximum and the minimum will be reversed.

Exam-style questions

14 a $A = \frac{1}{2}\times 5\times 10\sin 30^{\circ} = \frac{25}{2}$ (2 marks)

b $BD^2 = 5^2 + 10^2 - 2\times 5\times 10\cos 30^{\circ}$ (2 marks)

$BD = \sqrt{125 - 50\sqrt{3}}$ (1 mark)

$BD = \sqrt{25\left(5 - 2\sqrt{3}\right)}$ (1 mark)

$BD = 5\sqrt{5 - 2\sqrt{3}}$

c $\frac{\sin C\hat{D}B}{13} = \frac{\sin 45^{\circ}}{5\sqrt{5 - 2\sqrt{3}}}$ (2 marks)

$\sin C\hat{D}B = \frac{13\sqrt{2}}{10\sqrt{5 - 2\sqrt{3}}}$ (1 mark)

d The angle $C\hat{D}B$ can either be acute or obtuse (1 mark)

and the two possible values add up to 180º. (1 mark)

15 a $l = \sqrt{5^2 + 3^2} = 5.83$ cm (2 marks)

$S = 2\times\left(\pi\times 3\times 5.83\ldots\right) = 110$ cm^2 (2 marks)

b $\frac{2\times\frac{1}{3}\times\pi\times 3^2\times 5}{\pi\times 3.05^2\times 10.1}\times 100 = 31.9\%$ (2 marks)

16 $2\cos^2 x = \sin 2x \Rightarrow 2\cos^2 x - 2\sin x\cos x = 0$ (1 mark)

$2\cos x(\cos x - \sin x) = 0$ (1 mark)

$\cos x = 0,\ \ \cos x = \sin x$ (1 mark)

(or $\cos x = 0,\ \ \tan x = 1$)

$\cos x = 0 \Rightarrow x = \frac{\pi}{2},\ x = \frac{3\pi}{2}$ (1 mark)

$\tan x = 1 \Rightarrow x = \frac{\pi}{4},\ x = \frac{5\pi}{4}$ (1 mark)

17 a **i** $-1 \le y \le 3$ (1 mark)

ii 2 (1 mark)

b $a = -2$ (1 mark)

$b = \frac{2\pi}{2} = \pi$ (2 marks)

$c = 1$ (1 mark)

c $-2\cos \pi x + 1 = 0 \Rightarrow \cos \pi x = \frac{1}{2}$ (1 mark)

$\pi x \in \left\{-\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}\right\}$ (1 mark)

$x \in \left\{-\frac{1}{3}, \frac{1}{3}, \frac{5}{3}, \frac{7}{3}\right\}$ (1 mark)

18 a $\frac{\cos x}{1-\sin x} - \tan x = \frac{\cos x}{1-\sin x} - \frac{\sin x}{\cos x}$ (1 mark)

$= \frac{\cos^2 x - \sin x(1-\sin x)}{(1-\sin x)(\cos x)}$ (1 mark)

$= \frac{\cos^2 x + \sin^2 x - \sin x}{(1-\sin x)(\cos x)}$ (1 mark)

$= \frac{1-\sin x}{(1-\sin x)(\cos x)}$ (1 mark)

$= \frac{1}{\cos x}$ (1 mark)

$= \sec x$

b $\frac{\cos 2x}{1-\sin 2x} - \tan 2x = \sec 2x$

So $\sec 2x = \sqrt{2}$ (1 mark)

$\cos 2x = \frac{1}{\sqrt{2}}$ (1 mark)

$2x = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}$ (2 marks)

$x = \frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$ (2 marks)

19 a $\frac{dy}{dx} = \frac{1}{1+\left(\frac{1}{x}\right)^2} \times \left(-\frac{1}{x^2}\right)$ (2 marks)

$\frac{dy}{dx} = -\frac{1}{1+x^2}$ (1 mark)

b Valid attempt to apply product rule (1 mark)

$\frac{dy}{dx} = 2xe^{\arctan x} + \frac{x^2 e^{\arctan x}}{1+x^2}$ (3 marks)

$\left(\frac{dy}{dx} = e^{\arctan x}\left(2x + \frac{x^2}{1+x^2}\right)\right)$

20 Valid attempt at implicit differentiation (1 mark)

$(\cos y)\frac{dy}{dx} = -\sin x\left(\sec^2(\cos x)\right)$ (2 marks)

At $\left(\frac{\pi}{2}, 0\right)$: $(\cos 0)\frac{dy}{dx} = -\sin\frac{\pi}{2}\left(\sec^2\left(\cos\frac{\pi}{2}\right)\right)$ (1 mark)

$\frac{dy}{dx} = -\sec^2 0$ (1 mark)

$= -1$ (1 mark)

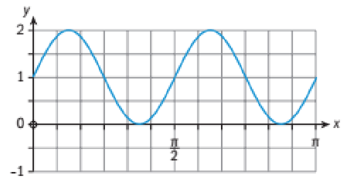
So gradient of normal is $-\frac{1}{(-1)} = 1$ (1 mark)

So equation is $y - 0 = 1\left(x - \frac{\pi}{2}\right)$, or $y = x - \frac{\pi}{2}$ (2 marks)

21 a $S(x) = \underbrace{\sin^2 2x + \cos^2 2x}_{1} + \underbrace{2\sin 2x\cos 2x}_{\sin 4x}$ (3 marks)

$= 1 + \sin 4x$

b

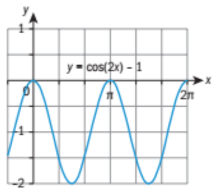


(1 mark) for correct shape, (1 mark) for 2 cycles, (1 mark) for correct max/min

c i $\frac{\pi}{2}$ (1 mark)

ii $0 \le y \le 2$ (1 mark)

d



(3 marks)

e **i** $k = 2$ (1 mark)

ii $p = -\frac{\pi}{4}$, $q = -2$ (2 marks)

22 a $D = \frac{22+12}{2}$ (1 mark)

$= 17$ (1 mark)

$A = \frac{22-12}{2}$ (1 mark)

$= 5$ (1 mark)

The period is $\frac{360}{B} = 24$ (1 mark)

Therefore $B = 15$ (1 mark)

So $T = 5\sin\left(15\left(t - C\right)\right) + 17$

At $\left(3, 12\right)$, $12 = 5\sin\left(15\left(3 - C\right)\right) + 17$ (1 mark)

$-1 = \sin\left(15\left(3 - C\right)\right)$

$15\left(3 - C\right) = -90$ (1 mark)

$C = 9$ (1 mark)

Therefore $T = 5\sin\left(15\left(t - 9\right)\right) + 17$

b Solving $T = 5\sin\left(15\left(t - 9\right)\right) + 17$ and $T = 20$ by GDC (1 mark)

Solutions are $T = 18.54$ and $T = 11.46$ (2 marks)

$18.54 - 11.46 = 7.08$ hours $\left(7 \text{ hours } 5 \text{ minutes}\right)$ (1 mark)

23 $V = \frac{1}{3}\pi r^2 h = \frac{50\pi r^2}{3}$ (1 mark)

$\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{100\pi r}{3}$ (2 marks)

$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t}$ (1 mark)

$2 = \frac{100\pi r}{3} \times \frac{\mathrm{d}r}{\mathrm{d}t}$ (1 mark)

$r = 0.4 \Rightarrow 2 = \frac{40\pi}{3} \times \frac{\mathrm{d}r}{\mathrm{d}t}$

$\Rightarrow \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{3}{20\pi}$ cm min^{-1} (1 mark)

7 Generalizing relationships: exponents, logarithms and integration

Skills check

1 $\dfrac{dy}{dx} = \dfrac{\cos x\left(3x^2-5\right)+\left(x^3-5x\right)\sin x}{\cos^2 x} = \dfrac{\cos x\left(3x^2-5\right)+\left(x^3-5x\right)\tan x}{\cos x}$

2 $3xy^2\dfrac{dy}{dx} + y^3 - 2y\sin x\cos x - \sin^2 x\dfrac{dy}{dx} = -y\sin x + \cos x\dfrac{dy}{dx}$

$\Rightarrow \left(3xy^2 - \sin^2 x - \cos x\right)\dfrac{dy}{dx} = y\left(\sin 2x - \sin x\right)$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{y\left(\sin 2x - \sin x\right)}{\left(3xy^2 - \sin^2 x - \cos x\right)}$

Exercise 7A

1 a $\int\left(-\dfrac{2}{3}x\right)dx = -\dfrac{2}{3}\cdot\dfrac{1}{2}x^2 + C = -\dfrac{1}{3}x^2 + C$

b $\int\dfrac{5}{4}x^3dx = \dfrac{5}{4}\cdot\dfrac{1}{4}x^4 + C = \dfrac{5}{16}x^4 + C$

c $\int 4x^{\frac{3}{2}}dx = 4\cdot\dfrac{2}{5}x^{\frac{5}{2}} + C = \dfrac{8}{5}x^{\frac{5}{2}} + C$

d $\int\dfrac{7}{2}x^{-\frac{1}{2}}dx = \dfrac{7}{2}\cdot 2x^{\frac{1}{2}} + C = 7x^{\frac{1}{2}} + C$

e $\int 2\sin\left(\dfrac{x}{2}\right)\cos\left(\dfrac{x}{2}\right)dx = \int\sin x dx = -\cos x + C$

2 a $\int\left(x^3+3x^2-4x+3\right)dx = \dfrac{1}{4}x^4 + x^3 - 2x^2 + 3x + C$

b $\int\left(x^4+4x^3-3x^{-3}+x^{-4}\right)dx = \dfrac{1}{5}x^5 + x^4 + \dfrac{3}{2}x^{-2} - \dfrac{1}{3}x^{-3} + C$

c $\left(x^2-x+2\right)^2 = x^4 - 2x^3 + 5x^2 - 4x + 4$

So,

$\int\left(x^2-x+2\right)^2dx = \int\left(x^4-2x^3+5x^2-4x+4\right)dx = \dfrac{1}{5}x^5 - \dfrac{1}{2}x^4 + \dfrac{5}{3}x^3 - 2x^2 + 4x + C$

d $\int(2-x)^3dx = \int\left(8-12x+6x^2-x^3\right)dx = 8x - 6x^2 + 2x^3 - \dfrac{1}{4}x^4 + C$

e $\int\left(x+\cos x-\dfrac{1}{\cos^2 x}\right)dx = \int\left(x+\cos x-\sec^2 x\right)dx$

$= \dfrac{1}{2}x^2 + \sin x - \tan x + C$

3 $y = f(x) = \int\left(3x^2-4\right)dx = x^3 - 4x + C$

$f(-1) = -1 + 4 + C = 2 \Rightarrow C = -1$

$\therefore y = f(x) = x^3 - 4x - 1$

4 $f(t) = \int\left(t - 2 + t^{-\frac{1}{2}}\right)dt = \frac{1}{2}t^2 - 2t + 2t^{\frac{1}{2}} + C$

$f(4) = 8 - 8 + 4 + C = 4 \Rightarrow C = 0$

$\therefore f(t) = \frac{1}{2}t^2 - 2t + 2t^{\frac{1}{2}}$

5 $y = \int(2x-1)^3\,dx = \int\left(8x^3 - 12x^2 + 6x - 1\right)dx$

$= 2x^4 - 4x^3 + 3x^2 - x + C$

At $x = -1,\ y = 2$

$\therefore 2 = 2 + 4 + 3 + 1 + C \Rightarrow C = -8$

$\Rightarrow y = 2x^4 - 4x^3 + 3x^2 - x - 8$

6 $\int\left(\sin\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right)\right)^2 dx = \int\left(\sin^2\left(\frac{\theta}{2}\right) + 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right)\right)d\theta$

$= \int\left(1 + 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right)d\theta = \int(1 + \sin\theta)\,d\theta$

$= \theta - \cos\theta + C$

7 $f(\theta) = \int(2 - 3\sin\theta)\,d\theta = 2\theta + 3\cos\theta + C$

$f(0) = 3 + C = -2 \Rightarrow C = -5$

So $f(\theta) = 2\theta + 3\cos\theta - 5$

8 **a** $f(x) = \int(3 - \cos x)dx = 3x - \sin x + C$

$f\left(\frac{\pi}{6}\right) = \frac{\pi}{2} - \frac{1}{2} + C = \frac{\pi}{2} \Rightarrow C = \frac{1}{2}$

$\therefore f(x) = 3x - \sin x + \frac{1}{2}$

b $f'(x) = 2\sec^2 x - 3\sin x$

$f(x) = \int\left(2\sec^2 x - 3\sin x\right)dx = 2\tan x + 3\cos x + C$

$f(0) = 3 + C = 4 \Rightarrow C = 1$

$\therefore f(x) = 2\tan x + 3\cos x + 1$

c $f'(x) = -2\cos x + C$

$f'\left(\frac{\pi}{4}\right) = -\sqrt{2} + C = 0 \Rightarrow C = \sqrt{2}$

$f(x) = -2\sin x + \sqrt{2}x + D$

$f(0) = D = 1$

$\therefore f(x) = -2\sin x + \sqrt{2}x + 1$

d $f'(x) = 2x + 3\sin x + C$

$f'(0) = C = 4$
$f(x) = x^2 - 3\cos x + 4x + D$
$f(0) = -3 + D = 5 \Rightarrow D = 8$
$\therefore f(x) = x^2 - 3\cos x + 4x + 8$

9 Let the velocity be $v(t)$ and the displacement be $s(t)$

$v(t) = \int a(t)\,dt = 9t^2 - 2t + C$
$v(0) = C = 1$
$\therefore v(t) = 9t^2 - 2t + 1$
$s(t) = \int v(t)\,dt = 3t^3 - t^2 + t + D$
$s(1) = 3 - 1 + 1 + D = 3 \Rightarrow D = 0$
$\therefore s(t) = 3t^3 - t^2 + t$

10 Let the velocity be $v(t)$ and the displacement be $s(t)$

$v(t) = \int a(t)\,dt = 6\sin t + C$
$v(0) = 0 + C = 0 \Rightarrow C = 0$
$\therefore v(t) = 6\sin t$
$s(t) = \int v(t)\,dt = -6\cos t + D$
$s\left(\frac{\pi}{3}\right) = -3 + D = 2 \Rightarrow D = 5$
$\therefore s(t) = -6\cos t + 5$

11 a $\int (1-3x)^6\,dx = -\frac{1}{7\cdot 3}(1-3x)^7 + C = -\frac{1}{21}(1-3x)^7 + C$

b $\int 3(4-x)^{\frac{1}{2}}\,dx = -3\cdot\frac{2}{3}(4-x)^{\frac{3}{2}} + C = -2(4-x)^{\frac{3}{2}} + C$

c $\int \left(3\cos(5x+2) + 4\sin(5x+2)\right)dx = \frac{3}{5}\sin(5x+2) - \frac{4}{5}\cos(5x+2) + C$

d $\int \left(2(2-3x)^{-\frac{1}{3}} + (1+2x)^{\frac{1}{3}}\right)dx = -2\cdot\frac{1}{3}\cdot\frac{3}{2}(2-3x)^{\frac{2}{3}} + \frac{1}{2}\cdot\frac{3}{4}(1+2x)^{\frac{4}{3}} + C$

$= -(2-3x)^{\frac{2}{3}} + \frac{3}{8}(1+2x)^{\frac{4}{3}} + C$

12 $f'(\theta) = 2\sin\left(2\theta + \frac{\pi}{2}\right)$

$f(\theta) = -\cos\left(2\theta + \frac{\pi}{2}\right) + C$
$f(0) = C = 1$
$\therefore f(\theta) = 1 - \cos\left(2\theta + \frac{\pi}{2}\right)$

Exercise 7B

1 Integral: $\int_0^4 5dx = [5x]_0^4 = 5(4-0) = 20$

Area of a rectangle: $5(4) = 20$

2 Integral: $\int_0^{10} |2x-5|\, dx = -\int_0^{2.5} (2x-5)\, dx + \int_{2.5}^{10} (2x-5)\, dx$

$= -\left[x^2 - 5x\right]_0^{2.5} + \left[x^2 - 5x\right]_{2.5}^{10} = -(6.25 - 12.5) + 0 + (100 - 50) - (6.25 - 12.5)$

$= 62.5$

Area: $\frac{1}{2}(5)(2.5) + \frac{1}{2}(7.5)(15) = 62.5$

3 Integral: $\int_{-3}^{3} (3 - |x|)dx = \int_{-3}^{0} (3+x)dx + \int_0^3 (3-x)\, dx$

$= \left[3x + \frac{x^2}{2}\right]_{-3}^{0} + \left[3x - \frac{x^2}{2}\right]_0^3$

$= 0 - \left(-9 + \frac{9}{2}\right) + \left(9 - \frac{9}{2}\right) - 0 = 9$

Area of triangle: $\frac{1}{2}bh = \frac{1}{2}(6)(3) = 9$

4 Integral: $\int_{-3}^{0} 4dx + \int_0^4 (4-x)\, dx = 4[x]_{-3}^{0} + \left[4x - \frac{x^2}{2}\right]_0^4$

$= 12 + (16 - 8) = 20$

Area of trapezium: $\left(\frac{a+b}{2}\right)h = \left(\frac{7+3}{2}\right)(4) = 20$

5 **a** $\int_{-3}^{1} (x+3)\, dx + \int_1^3 (6-2x)\, dx = \left[\frac{x^2}{2} + 3x\right]_{-3}^{1} + \left[6x - x^2\right]_1^3 = 12$

b The height of the triangle is 4 and the base length is 6 since the lines respectively intersect the x-axis at $x = -3$ and $x = 3$

$\therefore$ Area of triangle: $\frac{1}{2}bh = \frac{1}{2}(6)(4) = 12$

6 $\int_{-1}^{1} |x - x^3|\, dx = 2\int_0^1 (x - x^3)\, dx$ by symmetry as $f(x)$ is an odd function

$= 2\left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1 = 2\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2}$

7 $\int_{-1}^{6} |x^2 - 5x - 6|\, dx = -\int_{-1}^{6} (x^2 - 5x - 6)\, dx$

$$= -\left[\frac{1}{3}x^3 - \frac{5}{2}x^2 - 6x\right]_{-1}^{6} = -\left(\frac{1}{3}(216) - \frac{5}{2}(36) - 36\right) + \left(-\frac{1}{3} - \frac{5}{2} + 6\right)$$

$$= -(-54) + \left(\frac{-2 - 15 + 36}{6}\right) = 54 + \frac{19}{6} = \frac{343}{6}$$

8 $\int_1^4 \left|x^2 - 5x + 6\right| dx = \int_1^2 \left(x^2 - 5x + 6\right) dx - \int_2^3 \left(x^2 - 5x + 6\right) dx + \int_3^4 \left(x^2 - 5x + 6\right) dx$

$$= \left[\frac{x^3}{3} - \frac{5}{2}x^2 + 6x\right]_1^2 - \left[\frac{x^3}{3} - \frac{5}{2}x^2 + 6x\right]_2^3 + \left[\frac{x^3}{3} - \frac{5}{2}x^2 + 6x\right]_3^4$$

$$= \frac{11}{6}$$

9 When $\frac{x}{3} = n\pi - \frac{\pi}{2}$, $4\sin\left(\frac{x}{3} + \frac{\pi}{2}\right) = 4\sin\left(n\pi - \frac{\pi}{2} + \frac{\pi}{2}\right) = 4\sin(n\pi) = 0$

a $x = 3\pi\left(n - \frac{1}{2}\right)$ so take $n = 1 \Rightarrow x = \frac{3\pi}{2}$ for the first

and $n = 2 \Rightarrow x = \frac{9\pi}{2}$ for the second

b $x = 3\pi\left(n - \frac{1}{2}\right)$ so take $n = 0 \Rightarrow x = -\frac{3\pi}{2}$ for the first

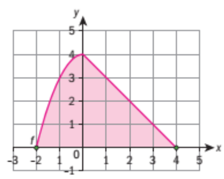
and $n = -1 \Rightarrow x = -\frac{9\pi}{2}$ for the second

c $\int_{-\frac{3\pi}{2}}^{\frac{9\pi}{2}} \left|4\sin\left(\frac{x}{3} + \frac{\pi}{2}\right)\right| dx$

$$= -\int_{-\frac{3\pi}{2}}^{\frac{3\pi}{2}} 4\sin\left(\frac{x}{3} + \frac{\pi}{2}\right) dx - \int_{\frac{3\pi}{2}}^{\frac{9\pi}{2}} 4\sin\left(\frac{x}{3} + \frac{\pi}{2}\right) dx$$

$$= 48$$

10



$$\int_{-2}^{0} \left(4 - x^2\right) dx + \int_0^4 (4 - x)\, dx$$

$$= \left[4x - \frac{x^3}{3}\right]_{-2}^{0} + \left[4x - \frac{x^2}{2}\right]_0^4$$

$$= \frac{16}{3} + 8 = \frac{40}{3}$$

Exercise 7C

1 a $(81)^{\frac{3}{4}} = \left(81^{\frac{1}{4}}\right)^3 = 3^3 = 27$

b $\left(\frac{8}{125}\right)^{-\frac{1}{3}} = \frac{125^{\frac{1}{3}}}{8^{\frac{1}{3}}} = \frac{5}{2}$

c $\left(-\frac{32}{243}\right)^{\frac{2}{5}} = \frac{(-2)^2}{(-3)^2} = \frac{4}{9}$

2 **a** $\left(\frac{a^{12}y^{-3}}{16y}\right)^{\frac{3}{4}} = \left(\frac{a^{12}}{16y^4}\right)^{\frac{3}{4}} = \frac{a^9}{8y^3}$

b $\frac{a^{-2}+2a^{-1}+1}{a^{-3}} = \frac{a^3\left(a^{-2}+2a^{-1}+1\right)}{a^3\left(a^{-3}\right)} = a+2a^2+a^3$

$= a\left(a^2+2a+1\right) = a\left(a+1\right)^2$

c $\frac{b^4 \times b^{-11}}{b^{-7}} = \frac{b^{-7}}{b^{-7}} = 1$

3 $\sqrt{9y^3} \div \sqrt[3]{8y^2} = \left(9y^3\right)^{\frac{1}{2}} \div \left(8y^2\right)^{\frac{1}{3}} = \frac{3}{2}y^{\frac{3}{2}-\frac{2}{3}} = \frac{3}{2}y^{\frac{5}{6}}$

Therefore, when $y = 64$,
$\sqrt{9y^3} \div \sqrt[3]{8y^2} = \frac{3}{2}(64)^{\frac{5}{6}} = \frac{3}{2}(32) = 48$

4 $\frac{\left(a^5b^2c^{-3}\right)\times\sqrt{a^{-3}b^3c}}{\sqrt{abc}} = \frac{a^{5-\frac{3}{2}}b^{2+\frac{3}{2}}c^{-3+\frac{1}{2}}}{a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}} = \frac{a^{\frac{7}{2}}b^{\frac{7}{2}}c^{-\frac{5}{2}}}{a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}} = \left(\frac{ab}{c}\right)^{\frac{6}{2}} = \left(\frac{ab}{c}\right)^3$

5 **a** $21\times3^{4n-1} - 7\times9^{2n} = 7\times3\times3^{4n-1} - 7\times\left(3^2\right)^{2n}$

$= 7\times3^{4n} - 7\times3^{4n} = 0$

b $48\times4^{2n-1} + 6\times2^{4n+1} = 3\times2^4\times\left(2^2\right)^{2n-1} + 3\times2\times2^{4n+1}$

$= 3\times2^{4n+2} + 3\times2^{4n+2} = 3\times2^{4n+3}$

6 **a** $2^{3x} - 2^{7x-2} = 2^{3x}\left(1-2^{4x-2}\right) = 0$

$2^{3x} \neq 0$ so $1-2^{4x-2} = 0 \Rightarrow 2^{4x} = 4 \Rightarrow x = \frac{1}{2}$

b $9^{1-2x} = \frac{1}{27^{x-4}} \Rightarrow \left(3^2\right)^{1-2x} = \frac{1}{\left(3^3\right)^{x-4}}$

$\therefore 3^{2-4x} = 3^{12-3x} \Rightarrow 3^{x+10} = 1 \Rightarrow x = -10$

c $9^x + 9 = 10\times3^x \Rightarrow 3^{2x} + 9 = 10\times3^x$

Let $y = 3^x$, then
$y^2 - 10y + 9 = (y-9)(y-1) = 0 \Rightarrow y = 9$ or $y = 1$
$y = 9 \Rightarrow x = 2$
$y = 1 \Rightarrow x = 0$

d $2^{x+2}+7=\dfrac{1}{2^{x-1}} \Rightarrow 2^{2x+1}+7\times 2^{x-1} \ =1 \Rightarrow 4\times 2^{2x}+7\times 2^{x}-2=0$

Let $y=2^x$, then
$4y^2+7y-2=0 \Rightarrow (4y-1)(y+2)=0$
$\therefore y=\dfrac{1}{4} \Rightarrow x=-2 \quad (y>0 \text{ so } y\neq -2)$

7 $300000r^{10}=500000 \Rightarrow r=\left(\dfrac{5}{3}\right)^{\frac{1}{10}}=1.0524\ldots$ so 5 %

8 **a** **i** $\dfrac{1}{6}\times\dfrac{48.76-37.21}{37.21}\times 100=5.17\% \quad (\text{to 3s.f.})$

ii $\dfrac{1}{6}\times\left(\dfrac{51.97-48.76}{48.76}\right)\times 100=1.10\% \quad (\text{to 3s.f.})$

iii The average percentage increase between December 2015 to December 2016 is equal to the average of the percentage increases from December 2015 to June 2016 and June 2016 to December 2016 (this is equal to 3.31% to 3s.f.)

b Individual Response.

c $(53.47-49.33)\div 30=0.138=14$ cents

9 Paloma: $2000(1.06)^{4\times 10}=20571.44 \ (\text{to the nearest hundredth})$

Concita: $1500(1.06)^{10}+500(1.03)^{10\times 12}=20041.77 \quad (\text{to the nearest hundredth})$

10 The amount Sureepan pays on the n^{th} year is given by $40000\times(1.075\times 0.5)^n$

So after four years he pays
$40000\times(1.075\times 0.5+\ldots+1.075^4\times 0.5^4)=42606.41$ Bhat
to the nearest hundredth

Exercise 7D

1 **a** $5=\log_3 243$

b $\log_{16} 2=\dfrac{1}{4}$

c $\log_q p=5$

d $-4=\log_{10}(0.0001)$

e $y=\log_x 11$

2 **a** $5^4=625$

b $64^{\frac{1}{2}}=8$

c $m^p=n$

d $b^0=1$

e $10^{-2} = 0.01$

3 **a** $\log_x 2 = 128 \Rightarrow 2 = x^{128} \Rightarrow x = 2^{\frac{1}{128}}$

b $\log_4 x = 3 \Rightarrow x = 4^3 = 64$

c $\log_x 8 = \dfrac{3}{4} \Rightarrow 8 = x^{\frac{3}{4}} \Rightarrow x = 8^{\frac{4}{3}} = 16$

d $\log_9 x = \dfrac{3}{2} \Rightarrow x = 9^{\frac{3}{2}} = 27$

e $\log_x 49 = 2 \Rightarrow 49 = x^2 \Rightarrow x = 7 \quad (x > 0)$

4 **a** $\log_a \dfrac{m}{n^2} = \log_a m - \log_a n^2 = \log_a m - 2\log_a n$

b $\log_a \left(\dfrac{\sqrt{m}}{n^3}\right)^{\frac{2}{3}} = \dfrac{2}{3}\log_a\left(\dfrac{\sqrt{m}}{n^3}\right) = \dfrac{2}{3}\left(\log_a \sqrt{m}\right) - \dfrac{2}{3}\log_a\left(n^3\right) = \dfrac{1}{3}\log_a m - 2\log_a n$

5 **a** $\log 6 - 3\log 2 + \log 40 = \log 6 - \log 8 + \log 40 = \log\left(\dfrac{40 \times 6}{8}\right) = \log 30$

b $\log_3 36 - 2 = \log_3 36 - \log_3 9 = \log_3 \dfrac{36}{9} = \log_3 4$

c $\dfrac{1}{4}\log_a m + \dfrac{3}{4}\log_a mn^2 = \log_a\left(m^{\frac{1}{4}} \cdot \left(mn^2\right)^{\frac{3}{4}}\right) = \log_a\left(mn^{\frac{3}{2}}\right)$

6 **a** $\log_2 12 - \log_2 48 - 3 = \log_2 \frac{12}{48} - 3 = \log_2 \frac{1}{4} - 3 = -2 - 3 = -5$

b $\dfrac{1}{4}\log_2 81 + \log_2 48 - \dfrac{2}{3}\log_2 27 = \log_2 3 + \log_2 48 - \log_2 9$

$= \log_2\left(\dfrac{3 \times 48}{9}\right) = \log_2 16 = 4$

7 **a** $4\log a = 3\log b \Rightarrow \log\left(a^4\right) = \log\left(b^3\right) \Rightarrow a^4 = b^3$

$\therefore a = b^{\frac{3}{4}}$

b $\log a = \log b - \log 2 \Rightarrow \log\left(\dfrac{2a}{b}\right) = 0 \Rightarrow \dfrac{2a}{b} = 1$

$\therefore a = \dfrac{b}{2}$

c $\log b = 1 - 4\log a \Rightarrow \log b = \log 10 - \log\left(a^4\right)$

$$\Rightarrow \log\left(a^4\right) = \log\left(\frac{10}{b}\right)$$

$$\therefore a^4 = \frac{10}{b}$$

$$\Rightarrow a = \left(\frac{10}{b}\right)^{\frac{1}{4}}$$

8 **a** $\log_2 3 \times \log_3 2 = \log_2 3 \times \dfrac{1}{\log_2 3} = 1$

b $\log_6 10 \times \log 36 = \dfrac{1}{\log 6} \times \left(2\log 6\right) = 2$

c $\log_4 3 \times \log_3 8 = \dfrac{1}{\log_3 4} \times \log_3 8 = \dfrac{1}{2\log_3 2} \times \left(3\log_3 2\right) = \dfrac{3}{2}$

d $\log_5 8 \div \log_{25} 8 = \dfrac{1}{\log_8 5} \div \dfrac{1}{\log_8 25} = \dfrac{1}{\log_8 5} \times \left(2\log_8 5\right) = 2$

e $\dfrac{1}{\log_3 6} + \dfrac{1}{\log_2 6} = \log_6 3 + \log_6 2 = \log_6 6 = 1$

f $\square$ $\log_5 40 - \dfrac{1}{2\log_{64} 5} = \log_5 40 - \dfrac{1}{2}\log_5 64 = \log_5 40 - \log_5 8$

$$= \log_5\left(\frac{40}{8}\right) = \log_5 5 = 1$$

9 **a** $x^{\log y} = \left(10^{\log x}\right)^{\log y} = 10^{\log x \log y} = \left(10^{\log y}\right)^{\log x} = y^{\log x}$

b $\dfrac{1}{\log_x xy} + \dfrac{1}{\log_y xy} = \log_{xy} x + \log_{xy} y = \log_{xy}\left(xy\right) = 1$

10 $p = \log_a x \Rightarrow \dfrac{1}{p} = \dfrac{1}{\log_a x} = \log_x a$

$$\therefore \log_x a = \frac{1}{p}$$

Similarly,

$$\frac{1}{q} = \frac{1}{\log_a y} = \log_y a \Rightarrow \log_y a = \frac{1}{q}$$

a $\log_{xy} a = \dfrac{1}{\log_a xy} = \dfrac{1}{\log_a x + \log_a y} = \dfrac{1}{p+q}$

b $\log_{\frac{x}{y}} a = \dfrac{1}{\log_a\left(\frac{x}{y}\right)} = \dfrac{1}{\log_a x - \log_a y} = \dfrac{1}{p-q}$

11 **a** $x = \log_3 10 = 2.10 \ \ (3\text{s.f.})$

b $5^{3x-1} = 12 \Rightarrow 3x - 1 = \log_5 12$

$$\therefore x = \frac{\log_5 12 + 1}{3} = 0.848 \ \text{(to 3s.f.)}$$

c $2^x \times 5^{x-1} = \frac{1}{5}(2 \times 5)^x = \frac{1}{5}10^x = 0.01$

$\Rightarrow 10^x = 0.05 \Rightarrow x = \log(0.05) = -1.30 \ \text{(3s.f.)}$

12 a $\log_5 x = 9\log_x 5 = \dfrac{9}{\log_5 x}$

$\Rightarrow (\log_5 x)^2 = 9 \Rightarrow \log_5 x = \pm 3$

$\therefore x = 5^3 = 125 \ \text{ or } \ x = 5^{-3} = \dfrac{1}{125}$

b $3\log_7 x + \log_7 49 = \log_7 (x^3) + 2 = 8$

$\therefore \log_7 (x^3) = 6$

$\Rightarrow x^3 = 7^6$

$\Rightarrow x = (7^6)^{\frac{1}{3}} = 7^2 = 49$

c $\log_4 x + \log_x 4 = \log_4 x + \dfrac{1}{\log_4 x} = 2$

$\therefore (\log_4 x)^2 - 2\log_4 x + 1 = 0$

$\Rightarrow (\log_4 x - 1)^2 = 0$

$\Rightarrow \log_4 x = 1$

$\Rightarrow x = 4$

13 $25^x - 6 \times 5^x - 16 = (5^x)^2 - 6 \times 5^x - 16 = 0$

$\Rightarrow (5^x - 8)(5^x + 2) = 0$

$5^x > 0 \Rightarrow 5^x = 8 \Rightarrow x = \log_5 8$

14 $\log_2 x + \dfrac{1}{\log_x 4} = \log_2 x + \dfrac{1}{2\log_x 2} = \log_2 x + \dfrac{1}{2}\log_2 x = 9$

$\Rightarrow \dfrac{3}{2}\log_2 x = 9$

$\Rightarrow \log_2 x = 6$

$\Rightarrow x = 2^6 = 64$

15 a $\log_5 x + 12\log_x 5 = \log_5 x + \dfrac{12}{\log_5 x} = 7$

$\Rightarrow (\log_5 x)^2 - 7\log_5 x + 12 = 0$

$\Rightarrow (\log_5 x - 4)(\log_5 x - 3) = 0$

$\therefore \log_5 x = 4 \ \text{ or } \ \log_5 x = 3$

so $x = 5^4 = 625 \ \text{ or } \ x = 5^3 = 125$

b $5 \times (7^x)^2 - 21 \times 7^x + 4 = 0$

$$\Rightarrow (5\times 7^x - 1)(7^x - 4) = 0$$

$$\Rightarrow 7^x = \frac{1}{5} \quad \text{or} \quad 7^x = 4$$

$$\Rightarrow x = \log_7\left(\frac{1}{5}\right) = -0.827 \quad \text{or} \quad x = \log_7 4 = 0.712$$

16 Using technology,□−3.42, 2.71

Eq: $4\times 9^x + 3\times 4^x = 13\times 6^x$
x=2.709511291
Lft=1668.602737
Rgt=1668.602737

17 a The first equation is equivalent to $2\log_5 x + 3\log_3 y = 16$

$$\therefore (2\log_5 x + 3\log_3 y) + 3(5\log_5 x - \log_3 y)$$
$$= 17\log_5 x = 16 + 3(6) = 34$$
$$\Rightarrow \log_5 x = 2 \Rightarrow x = 25$$
$$\therefore \log_3 y = 5\log_5 x - 6 = 4 \Rightarrow y = 81$$

b $3\log_a b = 1 \Rightarrow a = b^3$

$$\therefore ab = b^4 = 16 \Rightarrow b = 2 \quad (b > 0)$$
$$\Rightarrow a = 8$$

c $2m\log_4 16 = n \Rightarrow 4m = n$

$$\therefore 81^m + 3^n = 81^{\frac{n}{4}} + 3^n = 2\cdot 3^n = 54 \Rightarrow n = 3$$
$$\Rightarrow m = \frac{3}{4}$$

d $\log_4 x = \log_{16}(6x-9) = \frac{1}{2}\log_4(6x-9)$

$$\Rightarrow x^2 = 6x - 9 \Rightarrow (x-3)^2 = 0 \Rightarrow x = 3$$
$$\Rightarrow y = \log_4 3$$

18 $\dfrac{6\left(1-\left(\frac{2}{3}\right)^k\right)}{1-\frac{2}{3}} > 17.8$

$$\Rightarrow 1-\left(\frac{2}{3}\right)^k > \frac{17.8}{18}$$

$$\Rightarrow \left(\frac{2}{3}\right)^k < 1-\frac{17.8}{18}$$

$$\Rightarrow k\log\left(\frac{2}{3}\right) < \log\left(1-\frac{17.8}{18}\right)$$

$$\Rightarrow k > \frac{\log\left(1-\frac{17.8}{18}\right)}{\log\left(\frac{2}{3}\right)} = 11.10....$$

$$\Rightarrow k_{\text{min}} = 12$$

19 a $5+2\left[\frac{7}{8}\times 5+\left(\frac{7}{8}\right)^2\times 5\right] = \frac{685}{32} = 21.4\text{m}$ (to the nearest tenth of a metre)

b □ $5+2\times 5\times\left(\frac{7}{8}+\left(\frac{7}{8}\right)^2+\left(\frac{7}{8}\right)^3+\ldots+\left(\frac{7}{8}\right)^{k-1}\right)$

$$5+10\left(\frac{7}{8}\cdot\frac{1-\left(\frac{7}{8}\right)^{k-1}}{1-\frac{7}{8}}\right) = 5+10\cdot\frac{7}{8}\cdot\frac{1-\left(\frac{7}{8}\right)^{k-1}}{\frac{1}{8}} = 5+70\left(1-\left(\frac{7}{8}\right)^{k-1}\right)$$

c $5+70\left(1-\left(\frac{7}{8}\right)^{k-1}\right) < 39.5$

$$\Rightarrow 1-\left(\frac{7}{8}\right)^{k-1} < \frac{34.5}{70}$$

$$\Rightarrow \left(\frac{7}{8}\right)^{k-1} < 1-\frac{34.5}{70}$$

$$\Rightarrow (k-1) < \frac{\log\left(1-\frac{34.5}{70}\right)}{\log\left(\frac{7}{8}\right)}$$

$$\Rightarrow k < 1+\frac{\log\left(1-\frac{34.5}{70}\right)}{\log\left(\frac{7}{8}\right)} = 6.08\ldots$$

Therefore $k_{\text{max}} = 6$

20 $a = 5$

$$r = \frac{5.5}{5} = 1.1$$

$$S_n = 5\frac{1.1^n-1}{1.1-1} < 300$$

$$\Rightarrow 1.1^n < 6+1 = 7$$

$$\Rightarrow n < \frac{\log 7}{\log 1.1} = 20.42\ldots$$

So 20 experiments

Exercise 7E

1 Green: $a = \frac{1}{4}$

Red: $a = 2$

2 **a** $f(x+1) = ka^{x+1} = aka^x = af(x)$

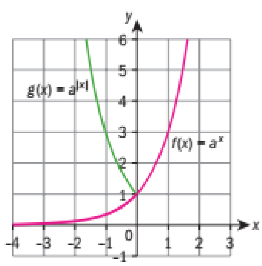
b $f(x+2) = ka^{x+2} = a^2ka^x = a^2f(x)$

c $f(x-1) = ka^{x-1} = a^{-1}ka^x = a^{-1}f(x)$

d Conjecture: $f(x+n) = a^nf(x)$

Proof: $f(x+n) = ka^{x+n} = a^nka^x = a^nf(x)$

3



4 $e^x + e^{-x} = 2 \Rightarrow e^{2x} - 2e^x + 1 = 0$

$\Rightarrow (e^x - 1)^2 = 0 \Rightarrow e^x = 1 \Rightarrow x = 0$

5 **a** A reflection in the line $x = 0$ (the y-axis)

$y = f(-x) = e^{-x}$

b A reflection in the line $y = 0$ (the x-axis)

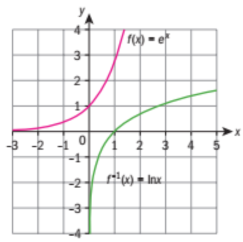
□ $y = -f(x) = -e^x$

c A reflection in the line $y = 0$ (the x – axis) followed by a reflection in the line $x = 0$ (the y – axis)

$y = -f(-x) = -e^{-x}$

6 The domain of f is $]-\infty, \infty[$, the range of f is $]0, \infty[$

The domain of f^{-1} is $]0, \infty[$, the range of f^{-1} is $]-\infty, \infty[$



7 $x = a^y \Rightarrow y = \log_a x$

So $f^{-1}(x) = \log_a x$

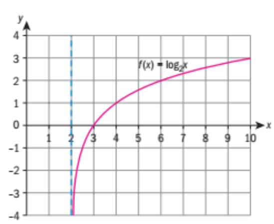
$f \circ f^{-1}(x) = a^{\log_a x} = x$ by direct substitution and definition of inverse function

8 **a** $f(x) = -\ln x$

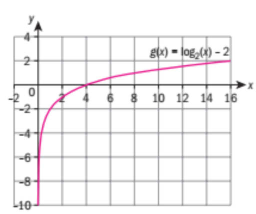
b $g(x) = |\ln x|$

c $h(x) = \ln|x|$

9

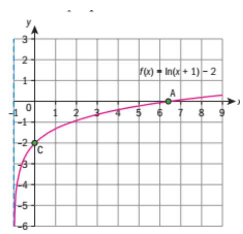


$y = \log_2(x-2)$ is a translation of $y = \log_2 x$ by 2 units in the positive x – direction

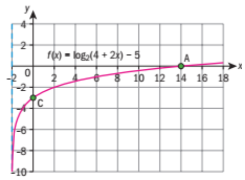


$y = (\log_2 x) - 2$ is a translation of $y = \log_2 x$ by 2 units in the negative y – direction

10 **a** Domain: $\{x \in \mathbb{R} : x > -1\}$, Asymptote: $x = -1$,



b Domain: $\{x \in \mathbb{R}, x > -2\}$, Asymptote: $x = -2$



Exercise 7F

1 **a** $\dfrac{dy}{dx} = 15e^{5x+4}$

b $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(3x^2\right)e^{3x^2} = 6xe^{3x^2}$

c $\dfrac{dy}{dx} = 4\left(\ln 5\right)5^{4x}$

d $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\cos x\right)e^{\cos x} = -\sin x\ e^{\cos x}$

e $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\sqrt{x}\right)\dfrac{1}{\sqrt{x}} = \left(\dfrac{1}{2\sqrt{x}}\right)\dfrac{1}{\sqrt{x}} = \dfrac{1}{2x}$

f $\dfrac{dy}{dx} = -\dfrac{5}{x}$

g $\dfrac{dy}{dx} = \dfrac{5}{5x+4}$

h $\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$

2 **a** $\dfrac{dy}{dx} = 3e^{2x+4} + 6xe^{2x+4} = 3e^{2x+4}\left(1+2x\right)$

b $y = \sqrt{x}e^{\sqrt[3]{x}} = x^{\frac{1}{2}}e^{x^{\frac{1}{3}}}$

$\dfrac{dy}{dx} = \left(\dfrac{1}{2}x^{-\frac{1}{2}}\right)e^{x^{\frac{1}{3}}} + x^{\frac{1}{2}}\left(\dfrac{1}{3}x^{-\frac{2}{3}}\right)e^{x^{\frac{1}{3}}} = \dfrac{1}{2}x^{-\frac{1}{2}}e^{x^{\frac{1}{3}}} + \dfrac{1}{3}x^{-\frac{1}{6}}e^{x^{\frac{1}{3}}}$

$= \dfrac{1}{6}e^{x^{\frac{1}{3}}}\left(3x^{-\frac{1}{2}} + 2x^{-\frac{1}{6}}\right)$

c $\dfrac{dy}{dx} = 3x^2\ln\left(2x+1\right) + x^3 \cdot \dfrac{2}{2x+1} = 3x^2\ln\left(2x+1\right) + \dfrac{2x^3}{2x+1}$

d $y = 3\sin xe^{-x}$

$\dfrac{dy}{dx} = 3\sin x\left(-e^{-x}\right) + 3\cos xe^{-x} = \dfrac{3\left(\cos x - \sin x\right)}{e^x}$

e $\dfrac{dy}{dx} = \left(2e^{2x}\right)\cdot\tan 3x + e^{2x}\cdot\left(3\sec^2 3x\right) = e^{2x}\left(2\tan 3x + 3\sec^2 3x\right)$

3 **a** $\frac{dy}{dx} = \frac{e^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right) - \sqrt{x}\left(\frac{1}{2\sqrt{x}}e^{\sqrt{x}}\right)}{e^{2\sqrt{x}}} = \frac{\frac{e^{\sqrt{x}}}{2\sqrt{x}}\left(1-\sqrt{x}\right)}{\left(e^{\sqrt{x}}\right)^2}$

$= \frac{1-\sqrt{x}}{2\sqrt{x}e^{\sqrt{x}}}$

b $\frac{dy}{dx} = \frac{\ln x \cdot \frac{1}{2\sqrt{x}} - \sqrt{x} \cdot \frac{1}{x}}{(\ln x)^2} = \frac{\frac{1}{2\sqrt{x}}(\ln x - 2)}{(\ln x)^2} = \frac{\ln x - 2}{2\sqrt{x}(\ln x)^2}$

c $\frac{dy}{dx} = \frac{(1+e^x)(-e^x)-(1-e^x)(e^x)}{(1+e^x)^2} = \frac{-e^x - e^{2x} - e^x + e^{2x}}{(1+e^x)^2} = -\frac{2e^x}{(1+e^x)^2}$

d $\frac{dy}{dx} = \frac{(1+x)\cdot\frac{1}{x} - \ln x}{(1+x)^2} = \frac{1+x(1-\ln x)}{x(1+x)^2}$

e $\frac{dy}{dx} = \frac{e^{-x}(1+e^x)-(x+e^x)(-e^{-x})}{(e^{-x})^2} = \frac{e^{-x}(1+e^x+x+e^x)}{(e^{-x})^2} = \frac{1+x+2e^x}{e^{-x}}$

4 **a** $y = e^{\sqrt{x}} + (\ln x)^{-1}$

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}e^{\sqrt{x}} - \frac{1}{x}(\ln x)^{-2} = \frac{e^{\sqrt{x}}}{2\sqrt{x}} - \frac{1}{x(\ln x)^2}$

b $y = x^x = e^{x\log x}$

$\frac{dy}{dx} = \frac{d}{dx}(x\log x)e^{x\log x} = (1+\log x)x^x$

5 **a** $y = \sqrt{1+e^x} = (1+e^x)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}e^x(1+e^x)^{-\frac{1}{2}}$

$\therefore$ At $(0, \sqrt{2})$

$\frac{dy}{dx} = \frac{1}{2}(2)^{-\frac{1}{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$

b $\frac{dy}{dx} = 2\cdot\frac{1}{1+x}\cdot\ln(1+x) = \frac{2\ln(1+x)}{1+x}$

At $\left(\frac{1}{2}, \left(\ln\frac{3}{2}\right)^2\right)$,

$\frac{dy}{dx} = \frac{2\ln\frac{3}{2}}{\frac{3}{2}} = \frac{4}{3}\ln\frac{3}{2}$

6 **a** $f'(x) = \frac{1}{x} - 1 = 0 \Rightarrow x = 1$

Thus there exists only solution to $f'(x) = 0$ so there is only turning point

b The turning point is located at $(1, f(1)) = (1, -1)$

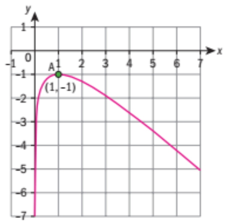
$f''(x) = -\frac{1}{x^2} < 0$ so the turning point must be a maximum

c Domain: $x > 0$

Range: $-\infty < f(x) \le -1$

d $x = 0$

e



7 **a** $f'(x) = e^x - 1 = 0 \Rightarrow x = 0$

Thus there exists only solution to $f'(x) = 0$ so there is only turning point

b The coordinates of the turning point is $(0, f(0)) = (0, 1)$

$f''(x) = e^x > 0$ so the turning point must be a minimum

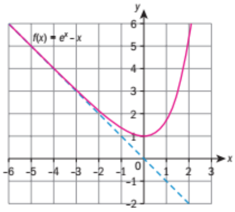
c Because there is a single turning point and it has just been shown that the point is a minimum

d Domain: $x \in \mathbb{R}$

Range: $f(x) \ge 1$

e $y = -x$

f



8 **a** Because $\ln|x|$ is not defined at $x = 0$

b $f(x) = 0 \Rightarrow x\ln|x| = 0$ or $x = 0$ by definition of the function

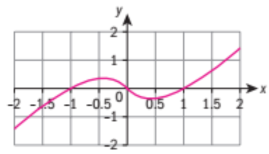
For $x \ne 0$, $x\ln|x| = 0 \Rightarrow |x| = 1 \Rightarrow x = \pm 1$

c For $x > 0,\ f'(x) = \ln x + 1 = 0 \Rightarrow x = e^{-1}$

For $x < 0$,
$f(x) = x\ln(-x)$
$\Rightarrow f'(x) = \ln(-x) + 1 = 0 \Rightarrow x = -e^{-1}$
So there are two turning points and these are located at $x = \pm e^{-1}$

d For $x > 0,\ f''(x) = \dfrac{1}{x}$
$\therefore f''(e^{-1}) = \dfrac{1}{e^{-1}} > 0$
so there is a minimum at $(e^{-1}, -e^{-1})$
and by symmetry, there is a maximum at $(-e^{-1}, e^{-1})$

e



9 a $f'(x) = 2xe^x + x^2e^x = xe^x(2 + x)$

$\therefore f'(x) = 0 \Rightarrow x = 0$ or $x = -2$ so there are two turning points

$f''(x) = 2e^x + 2xe^x + 2xe^x + x^2e^x = (2 + 4x + x^2)e^x$
$\therefore f''(x) = 0 \Rightarrow x^2 + 4x + 2 = 0 \Rightarrow x = -2 \pm \sqrt{2}$
So there are two points of inflection

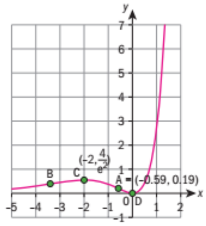
b The turning points are located at $(0,0)$ and $(-2, 4e^{-2})$

At $(0,0)$, $f''(0) = 2 > 0$ so this is a minimum
At $(-2, 4e^{-2})$, $f''(-2) = -2e^{-2} < 0$ so this is a maximum

c $x = -2 \pm \sqrt{2}$

d $y = 0$

e



f $f'(1) = 3e$

$\therefore y - f(1) = f'(1)(x-1)$
$\Rightarrow y - e = 3e(x-1)$
$\Rightarrow y = 3ex - 2e = e(3x-2)$

g $x-$intercept is located at $\left(\frac{2}{3}, 0\right)$

$y-$intercept is located at $(0, -2e)$

$\therefore$ The area of the triangle is given by $\frac{1}{2}(2e)\left(\frac{2}{3}\right) = \frac{2e}{3}$

10 a $f'(x) = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x)$

$f'(x) = 0 \Rightarrow e^x(\sin x + \cos x) = 0$
$\Rightarrow \tan x = -1 \quad (e^x \neq 0)$
so in the range $-\frac{3\pi}{2} < x < \frac{3\pi}{2}$, there are three roots are located at
$x = -\frac{5\pi}{4}$ and $x = -\frac{\pi}{4}$ and $x = \frac{3\pi}{4}$
$f''(x) = e^x(\sin x + \cos x) + e^x(\cos x - \sin x) = 2e^x \cos x$
$\therefore f''(x) = 0 \Rightarrow \cos x = 0 \quad (e^x \neq 0)$

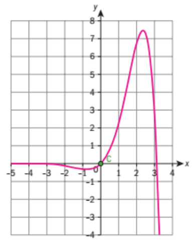
Therefore there are two points of inflexion, at $x = \pm\frac{\pi}{2}$

b $f''\left(-\frac{5\pi}{4}\right) = -\sqrt{2}e^{-\frac{5\pi}{4}} < 0$ so maximum at $x = -\frac{5\pi}{4}$

$f''\left(-\frac{\pi}{4}\right) = \sqrt{2}e^{-\frac{\pi}{4}} > 0$ so minimum at $x = -\frac{\pi}{4}$

$f''\left(\frac{3\pi}{4}\right) = -\sqrt{2}e^{\frac{3\pi}{4}} < 0$ so maximum at $x = \frac{3\pi}{4}$

c $\left(\frac{\pi}{2}, e^{\frac{\pi}{2}}\right)$ and $\left(-\frac{\pi}{2}, e^{-\frac{\pi}{2}}\right)$



d $f'\left(\frac{\pi}{2}\right) = e^{\frac{\pi}{2}}$ so the normal at this point has gradient $-e^{-\frac{\pi}{2}}$

$f'\left(-\frac{\pi}{2}\right) = -e^{-\frac{\pi}{2}}$ so this is parallel to the normal at $x = \frac{\pi}{2}$

Exercise 7G

1 **a** $\int\left(x^3-\sec^2 x\right)dx=\frac{x^4}{4}-\tan x+C$

b $\int\left(3e^x+\frac{1}{2x}+\sin x\right)dx=3e^x+\frac{1}{2}\ln x-\cos x+C$

c $\int 2\sin x\cos x dx=\int\sin 2x dx=-\frac{1}{2}\cos 2x+C$

d $\int\left(\tan^2 3x+1\right)dx=\int\sec^2 3x dx=\frac{1}{3}\tan 3x+C$

e $\int\left(\frac{3}{2x}+3^x\right)dx=\frac{3}{2}\ln x+\frac{1}{\ln 3}3^x$

f $\int\left(\frac{3}{1-3x}\right)dx=-\ln\left|1-3x\right|+C$

g $\int\left(\frac{x}{x-1}+\cos x\right)dx=\int\left(\frac{x-1+1}{x-1}+\cos x\right)dx$

$=\int\left(1+\frac{1}{x-1}+\cos x\right)dx=x+\ln\left|x-1\right|+\sin x+C$

h $\int\left(\left(\cos x\right)e^{\sin x}-\frac{2}{x}\right)dx=\int\left(\frac{d}{dx}\left(e^{\sin x}\right)-\frac{2}{x}\right)dx=e^{\sin x}-2\ln\left|x\right|+C$

i $\int\left(\frac{e^{\sqrt{x}}}{2\sqrt{x}}-4\sin x\cos x\right)dx=\int\left(\frac{d}{dx}\left(e^{\sqrt{x}}\right)-2\sin 2x\right)dx$

$=e^{\sqrt{x}}+\cos 2x+C$

2 **a** $f\left(x\right)=\int\left(2x-\sin 4x\right)dx=x^2+\frac{1}{4}\cos 4x+C$

$f\left(0\right)==\frac{1}{4}+C=0\Rightarrow C=-\frac{1}{4}$

$\therefore f\left(x\right)=x^2+\frac{1}{4}\cos 4x-\frac{1}{4}$

b $f\left(x\right)=\int\left(x^2+e^{-x}+\sec^2 x\right)dx=\frac{1}{3}x^3-e^{-x}+\tan x+C$

$f\left(0\right)=-1+C=-1\Rightarrow C=0$

$\therefore f\left(x\right)=\frac{1}{3}x^3-e^{-x}+\tan x$

c $f\left(x\right)=\int\left(\frac{3}{2x-5}-3x^2-3e^{x-3}\right)dx=\frac{3}{2}\ln\left(2x-5\right)-x^3-3e^{x-3}+C$

$f\left(3\right)=0-27-3+C=-25\Rightarrow C=5$

$\therefore f\left(x\right)=\frac{3}{2}\ln\left(2x-5\right)-x^3-3e^{x-3}+5$

3 **a** $f''\left(x\right)=2\cos^2 x-1=\cos 2x$

$$f'(x) = \frac{1}{2}\sin 2x + C$$

$$f'\left(\frac{\pi}{2}\right) = C = 1 \Rightarrow f'(x) = \frac{1}{2}\sin 2x + 1$$

$$f(x) = -\frac{1}{4}\cos 2x + x + D$$

$$f(0) = -\frac{1}{4} + D = \frac{3}{4} \Rightarrow D = 1$$

$$\therefore f(x) = -\frac{1}{4}\cos 2x + x + 1$$

b $f'(x) = \frac{1}{2}e^{2x-1} + \frac{1}{2}\cos(1-2x) + C$

$$f'\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{1}{2} + C = \frac{1}{2} \Rightarrow C = -\frac{1}{2}$$

$$\therefore f'(x) = \frac{1}{2}e^{2x-1} + \frac{1}{2}\cos(1-2x) - \frac{1}{2}$$

$$f(x) = \frac{1}{4}e^{2x-1} - \frac{1}{4}\sin(1-2x) - \frac{x}{2} + D$$

$$f\left(\frac{1}{2}\right) = \frac{1}{4} - \frac{1}{4} + D = \frac{1}{2} \Rightarrow D = \frac{1}{2}$$

$$\therefore f(x) = \frac{1}{4}e^{2x-1} - \frac{1}{4}\sin(1-2x) - \frac{x}{2} + \frac{1}{2}$$

4 a $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(x - \sec^2 x)dx = \left[\frac{x^2}{2} - \tan x\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \left(\frac{\pi^2}{32} - 1 - \frac{\pi^2}{32} - 1\right) = -2$

b $\int_0^{\frac{\pi}{3}}(e^{3x-2} + \sin x)dx = \left[\frac{1}{3}e^{3x-2} - \cos x\right]_0^{\frac{\pi}{3}}$

$$\left(\frac{1}{3}e^{\pi-2} - \frac{1}{2}\right) - \left(\frac{1}{3}e^{-2} - 1\right) = \frac{1}{3}e^{-2}(e^{\pi} - 1) + \frac{1}{2}$$

c $\int_1^2\left(\frac{1}{2^x} + x^2 - e^{2x}\right)dx = \left[-\frac{1}{\ln 2}2^{-x} + \frac{1}{3}x^3 - \frac{1}{2}e^{2x}\right]_1^2$

$$= \left(-\frac{1}{4\ln 2} + \frac{8}{3} - \frac{1}{2}e^4\right) - \left(-\frac{1}{2\ln 2} + \frac{1}{3} - \frac{1}{2}e^2\right)$$

$$= \frac{1}{\ln 16} + \frac{e^2}{2}(1 - e^2) + \frac{7}{3}$$

d $\int_0^1\left(3^x - \frac{6}{1-3x}\right)dx = \left[\frac{1}{\ln 3}3^x + 2\ln|1-3x|\right]_0^1 = \left(\frac{3}{\ln 3} + 2\ln 2\right) - \frac{1}{\ln 3}$

$$= 2\left(\frac{1}{\ln 3} + \ln 2\right) = \frac{2}{\ln 3} + \ln 4$$

e $\int_0^{\frac{\pi}{2}}\left(\sec^2\left(\frac{x}{2}\right) + xe^{x^2}\right)dx = \left[2\tan\left(\frac{x}{2}\right) + \frac{1}{2}e^{x^2}\right]_0^{\frac{\pi}{2}}$

$$= \left(2 + \frac{1}{2}e^{\frac{\pi^2}{4}}\right) - \frac{1}{2} = \frac{3}{2} + \frac{1}{2}e^{\frac{\pi^2}{4}} = \frac{1}{2}\left(1 + 3e^{\frac{\pi^2}{4}}\right)$$

f $\int_2^4 \frac{2x}{x-1}dx = 2\int_2^4 \frac{x-1+1}{x-1}dx = 2\int_2^4 \left(1+\frac{1}{x-1}\right)dx$

$= 2\left[x+\ln(x-1)\right]_2^4 = 2(4+\ln 3-2) = 2(2+\ln 3)$

g $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{\cos 6x}{\cos 3x+\sin 3x}\right)dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{\cos^2 3x-\sin^2 3x}{\cos 3x+\sin 3x}\right)dx$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\cos 3x-\sin 3x)dx = \left[\frac{1}{3}\sin 3x+\frac{1}{3}\cos 3x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= -\frac{1}{3}-\left(\frac{1}{3}\right) = -\frac{2}{3}$

h $\int_0^{\frac{\pi}{4}} \left(\frac{\sin 2x}{\cos x}\right)dx = 2\int_0^{\frac{\pi}{4}} \sin x dx = 2\left[-\cos x\right]_0^{\frac{\pi}{4}} = 2-\sqrt{2}$

i $\int_0^{\frac{\pi}{6}} \left(2\sin^2 x\cot x\right)dx = \int_0^{\frac{\pi}{6}} \sin 2x dx = -\frac{1}{2}\left[\cos 2x\right]_0^{\frac{\pi}{6}} = -\frac{1}{2}\left(\frac{1}{2}-1\right) = \frac{1}{4}$

j $\int_0^{\frac{1}{2}} \left(2^x+\frac{2}{1-x}\right)dx = \left[\frac{1}{\ln 2}2^x-2\ln(1-x)\right]_0^{\frac{1}{2}} = \frac{\sqrt{2}}{\ln 2}-2\ln\frac{1}{2}-\frac{1}{\ln 2}$

$= \frac{1}{\ln 2}\left(\sqrt{2}-1\right)+\ln 4$

5 $f(x) = \int \tan x dx = -\ln(\cos x)+C$

$f(0) = C = 0$

$\therefore f(x) = -\ln(\cos x)$

Range: $f(x) \geq 0$ $\left(f(x) \in \square\right)$

$f'\left(\frac{\pi}{4}\right) = 1$ so the gradient of the tangent here is 1 and the gradient of the normal is -1

Tangent: $y-\ln\sqrt{2} = \left(x-\frac{\pi}{4}\right) \Rightarrow y = x+\ln\sqrt{2}-\frac{\pi}{4}$

Normal: $y-\ln\sqrt{2} = -\left(x-\frac{\pi}{4}\right) \Rightarrow y = -x+\ln\sqrt{2}+\frac{\pi}{4}$

Base of triangle (along $y-$axis): $\left(\ln\sqrt{2}+\frac{\pi}{4}\right)-\left(\ln\sqrt{2}-\frac{\pi}{4}\right) = \frac{\pi}{2}$

Height of traingle: $\frac{\pi}{4}$

so area is $\frac{1}{2}\left(\frac{\pi}{2}\right)\left(\frac{\pi}{4}\right) = \frac{\pi^2}{16}$

6 $f(x) = \frac{x+9}{2x^2+x-3} = \frac{x+9}{(2x+3)(x-1)} = \frac{A}{2x+3}+\frac{B}{x-1}$

$\therefore x+9=A(x-1)+B(2x+3)$

Set $x=1$: $\quad 10=B(5)\Rightarrow B=2$

Compare coefficients of x: $\quad 1=A+2B=A+4\Rightarrow A=-3$

$\therefore f(x)=\dfrac{x+9}{2x^2+x-3}=\dfrac{2}{x-1}-\dfrac{3}{2x+3}$

$\Rightarrow \displaystyle\int \frac{x+9}{2x^2+x-3}dx=\int\left(\frac{2}{x-1}-\frac{3}{2x+3}\right)dx$

$=2\ln|x-1|-\dfrac{3}{2}\ln|2x+3|+C$

7 $\dfrac{1}{x^2-1}=\dfrac{1}{(x+1)(x-1)}=\dfrac{1}{2}\left(\dfrac{1}{x-1}-\dfrac{1}{x+1}\right)$

$\therefore \displaystyle\int \frac{1}{x^2-1}dx=\frac{1}{2}\int\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx=\frac{1}{2}\left(\ln(x-1)-\ln(x+1)\right)+C$

$=\dfrac{1}{2}\ln\left|\dfrac{x-1}{x+1}\right|+C$

8 $f(x)=\dfrac{5x+9}{x^2-9}=\dfrac{5x+9}{(x+3)(x-3)}=\dfrac{A}{x+3}+\dfrac{B}{x-3}$

$\therefore 5x+9=A(x-3)+B(x+3)$

Set $x=3$: $\quad 24=6B\Rightarrow B=4$

Set $x=-3$: $\quad -6=-6A\Rightarrow A=1$

$\therefore f(x)=\dfrac{5x+9}{x^2-9}=\dfrac{1}{x+3}+\dfrac{4}{x-3}$

$\Rightarrow \displaystyle\int\left(\frac{5x+9}{x^2-9}\right)dx=\int\left(\frac{1}{x+3}+\frac{4}{x-3}\right)dx=\ln(x+3)+4\ln(x-3)+C$

9 $\dfrac{1-2x}{x+x^2}=\dfrac{1-2x}{x(x+1)}=\dfrac{A}{x}+\dfrac{B}{x+1}$

$\therefore 1-2x=A(x+1)+Bx$

Set $x=0$: $\quad 1=A\Rightarrow A=1$

Set $x=-1$: $\quad 3=-B\Rightarrow B=-3$

$\therefore \dfrac{1-2x}{x(x+1)}=\dfrac{1}{x}-\dfrac{3}{x+1}$

$\Rightarrow \displaystyle\int_{\frac{1}{2}}^{1}\frac{1-2x}{x+x^2}dx=\int_{\frac{1}{2}}^{1}\left(\frac{1}{x}-\frac{3}{x+1}\right)dx=\left[\ln x-3\ln(x+1)\right]_{\frac{1}{2}}^{1}$

$=(0-3\ln 2)-\left(\ln\dfrac{1}{2}-3\ln\dfrac{3}{2}\right)=\ln\left(\dfrac{1}{8}\cdot 2\cdot\dfrac{27}{8}\right)=\ln\dfrac{27}{32}$

10 $\displaystyle\int_0^{\frac{1}{2}}\left(\frac{2+3x-x^2}{1-x^2}\right)dx=\int_0^{\frac{1}{2}}\left(\frac{1-x^2+1+3x}{1-x^2}\right)dx=\int_0^{\frac{1}{2}}\left(1+\frac{1+3x}{1-x^2}\right)dx$

$$= \frac{1}{2} + \int_0^{\frac{1}{2}} \frac{1+3x}{1-x^2}\,dx$$

$$\frac{1+3x}{1-x^2} = \frac{1+3x}{(1-x)(1+x)} = \frac{A}{1-x} + \frac{B}{1+x}$$

$$\Rightarrow 1+3x = A(1+x) + B(1-x)$$

Set $x = 1$: $4 = 2A \Rightarrow A = 2$

Set $x = -1$: $-2 = 2B \Rightarrow B = -1$

$$\therefore \frac{1+3x}{1-x^2} = \frac{2}{1-x} - \frac{1}{1+x}$$

So

$$\int_0^{\frac{1}{2}} \left(\frac{2+3x-x^2}{1-x^2}\right) dx = \frac{1}{2} + \int_0^{\frac{1}{2}} \left(\frac{2}{1-x} - \frac{1}{1+x}\right) dx$$

$$= \frac{1}{2} + \left[-2\ln(1-x) - \ln(1+x)\right]_0^{\frac{1}{2}} = \frac{1}{2} + \left(-2\ln\frac{1}{2} - \ln\frac{3}{2}\right) - 0$$

$$= \frac{1}{2} + \ln\left(4 \cdot \frac{2}{3}\right) = \frac{1}{2} + \ln\frac{8}{3}$$

Exercise 7H

1 Let $u = 3x^2 + 4 \Rightarrow du = 6x\,dx$

$$\therefore \int 6x\sqrt{3x^2+4}\,dx = \int u^{\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(3x^2+4)^{\frac{3}{2}} + C$$

2 Let $u = x^3 \Rightarrow du = 3x^2\,dx$

$$\therefore \int 3x^2 \cos x^3\,dx = \int \cos u\,du = \sin u + C = \sin(x^3) + C$$

3 Let $u = 2 + x - x^2 \Rightarrow du = (1-2x)\,dx$

$$\therefore \int (1-2x)e^{2+x-x^2}\,dx = \int e^u\,du = e^u + C = e^{2+x-x^2} + C$$

4 Let $u = \cos 2x \Rightarrow du = -2\sin 2x\,dx$

$$\therefore \int 2\sin 2x e^{\cos 2x}\,dx = -\int e^u\,du = -e^u + C = -e^{\cos 2x} + C$$

5 Let $u = 3^x \Rightarrow du = 3^x \ln 3\,dx$

$$\therefore \int 3^x \ln 3 \sin 3^x\,dx = \int \sin u\,du = -\cos u + C = -\cos(3^x) + C$$

6 Let $u = x^{\frac{3}{2}} \Rightarrow du = \frac{3}{2}x^{\frac{1}{2}}\,dx$

$$\therefore \int \frac{3}{2}\sqrt{x}e^{\sqrt{x^3}}\,dx = \int e^u\,du = e^u + C = e^{\sqrt{x^3}} + C$$

7 Let $u = x - 1 \Rightarrow du = dx$

$$\therefore \int 2x\sqrt{x-1}dx = \int 2(u+1)\sqrt{u}du = \int\left(2u^{\frac{3}{2}} + 2u^{\frac{1}{2}}\right)du$$

$$= 2\cdot\frac{2}{5}u^{\frac{5}{2}} + 2\cdot\frac{2}{3}u^{\frac{3}{2}} + C = \frac{4}{5}u^{\frac{5}{2}} + \frac{4}{3}u^{\frac{3}{2}} + C$$

$$= \frac{4}{5}(x-1)^{\frac{5}{2}} + \frac{4}{3}(x-1)^{\frac{3}{2}} + C$$

$$= \frac{4}{15}(x-1)^{\frac{3}{2}}\left(3(x-1)+5\right) + C$$

$$= \frac{4}{15}(x-1)^{\frac{3}{2}}(3x+2) + C$$

8 Let $u = x+1 \Rightarrow du = dx$

$$\int(1-x)\sqrt{1+x}dx = \int(2-u)\sqrt{u}du = \int\left(2u^{\frac{1}{2}} - u^{\frac{3}{2}}\right)du$$

$$= 2\cdot\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}} + C = \frac{4}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}} + C$$

$$= \frac{4}{3}(x+1)^{\frac{3}{2}} - \frac{2}{5}(x+1)^{\frac{5}{2}} + C$$

$$= \frac{2}{15}(x+1)^{\frac{3}{2}}\left(10-3(x+1)\right) + C$$

$$= \frac{2}{15}(x+1)^{\frac{3}{2}}(7-3x) + C$$

9 Let $u = x^3 - \frac{3}{2}x^2 \Rightarrow du = \left(3x^2 - 3x\right)dx$

$$\int\left(x^2+x\right)\sec^2\left(x^3 - \frac{3}{2}x^2\right)dx = \frac{1}{3}\int\sec^2 u du$$

$$= \frac{1}{3}\tan u + C = \frac{1}{3}\tan\left(x^3 - \frac{3}{2}x^2\right) + C$$

10 Let $u = \cos 2x \Rightarrow du = -2\sin 2x\ dx$

$$\int\sin 2x\left(2^{\cos 2x}\right)dx = -\frac{1}{2}\int 2^u du = -\frac{1}{2}\cdot\frac{1}{\ln 2}2^u + C$$

$$= -\frac{1}{2\ln 2}2^{\cos 2x} + C$$

11 Let $u = 1-x^2 \Rightarrow du = -2x\ dx$

$$\int x\sqrt{1-x^2}dx = -\frac{1}{2}\int u^{\frac{1}{2}}du = -\frac{1}{3}u^{\frac{3}{2}} + C = -\frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}} + C$$

12 Let $u = 1-x \Rightarrow du = -dx$

$$\int x^2\sqrt{1-x}dx = -\int(1-u)^2u^{\frac{1}{2}}du = -\int\left(u^{\frac{1}{2}} - 2u^{\frac{3}{2}} + u^{\frac{5}{2}}\right)du$$

$$= -\frac{2}{3}u^{\frac{3}{2}} + \frac{4}{5}u^{\frac{5}{2}} - \frac{2}{7}u^{\frac{7}{2}} + C$$

$$= -\frac{2}{3}(1-x)^{\frac{3}{2}} + \frac{4}{5}(1-x)^{\frac{5}{2}} - \frac{2}{7}(1-x)^{\frac{7}{2}} + C$$

$$= -\frac{2}{105}(1-x)^{\frac{3}{2}}\left(35 - 42(1-x) + 15(1-x)^2\right) + C$$

$$= -\frac{2}{105}(1-x)^{\frac{3}{2}}\left(8 + 12x + 15x^2\right) + C$$

13 Let $u = \sqrt{1-x} \Rightarrow du = -\dfrac{1}{2\sqrt{1-x}}dx$

$$\int \frac{x^2}{\sqrt{1-x}}dx = -2\int \left(1-u^2\right)^2 du = -2\int \left(1-2u^2+u^4\right)du$$
$$= -2\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right) + C$$
$$= -2(1-x)^{\frac{1}{2}} + \frac{4}{3}(1-x)^{\frac{3}{2}} - \frac{2}{5}(1-x)^{\frac{5}{2}} + C$$
$$= -\frac{2}{15}(1-x)^{\frac{1}{2}}\left(15 - 10(1-x) + 3(1-x)^2\right) + C\,.$$
$$= -\frac{2}{15}(1-x)^{\frac{1}{2}}\left(3x^2+4x+8\right) + C$$

14 Let $u = 1+x \Rightarrow du = dx$

$$\int x(1+x)^5\,dx = \int (u-1)u^5 du = \frac{u^7}{7} - \frac{u^6}{6} + C$$
$$= \frac{1}{7}(1+x)^7 - \frac{1}{6}(1+x)^6 + C$$
$$= -\frac{1}{42}(1+x)^6\left(-6(1+x)+7\right) + C$$
$$= -\frac{1}{42}(1+x)^6(1-6x) + C$$

15 Let $u = 1-\cos x \Rightarrow du = \sin x\ dx$

$$\int \left(\frac{\sin x}{1-\cos x}\right)dx = \int u^{-1}du = \ln(u) + C$$
$$= \ln(1-\cos x) + C$$

16 Let $u = e^{5-x} \Rightarrow du = -e^{5-x}dx$

$$\int e^{5-x}\sqrt{e^{5-x}}dx = -\int u^{\frac{1}{2}}du = -\frac{2}{3}u^{\frac{3}{2}} + C = -\frac{2}{3}e^{\frac{3}{2}(5-x)} + C$$

17 Let $u = 3-x \Rightarrow du = -dx$

$$\int \left(\frac{x}{3-x}\right)^2 dx = -\int \left(\frac{3-u}{u}\right)^2 du = -\int \left(\frac{9}{u^2} - \frac{6}{u} + 1\right)du$$
$$= 9u^{-1} + 6\ln u - u + C$$
$$= \frac{9}{3-x} + 6\ln(3-x) + x - 3 + C$$

Note it is permissible to incorporate the constant -3 into the arbitrary constant

18 Let $u = 2x-3 \Rightarrow x = \dfrac{u+3}{2}$ and $du = 2dx$

$$\int (1-x)\sqrt{2x+3}dx = -\frac{1}{4}\int (u+1)u^{\frac{1}{2}}du$$
$$= -\frac{1}{4}\int \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right)du = -\frac{1}{10}u^{\frac{5}{2}} - \frac{1}{6}u^{\frac{3}{2}} + C$$
$$= -\frac{1}{10}(2x+3)^{\frac{5}{2}} - \frac{1}{6}(2x+3)^{\frac{3}{2}} + C \quad .$$
$$= -\frac{1}{30}(2x+3)^{\frac{3}{2}}\left(3(2x+3)+5\right) + C$$
$$= -\frac{1}{15}(2x-3)^{\frac{3}{2}}(3x-7) + C$$

19 Let $u = 1 - x \Rightarrow du = -dx$

$$\int \frac{3(x-4)}{(1-x)^2}dx = -\int \frac{3(-u-3)}{u^2}du = 3\int \left(\frac{1}{u} + \frac{3}{u^2}\right)du$$
$$= 3\ln|u| - 9u^{-1} + C$$
$$= 3\ln|1-x| - \frac{9}{1-x} + C$$

20 Let $u = 2 + \tan x \Rightarrow du = \sec^2 x\ dx$

$$\int \left(\frac{\sec^2 x}{2+\tan x}\right)dx = \ln|u| + C = \ln|2+\tan x| + C$$

Exercise 7I

1 Let $u = x - 1 \Rightarrow du = dx$

Limits: $x = 2 \Rightarrow u = 1, \quad x = 5 \Rightarrow u = 4$

$$\int_2^5 \frac{x}{\sqrt{x-1}}dx = \int_1^4 \frac{u+1}{\sqrt{u}}du = \int_1^4 \left(u^{\frac{1}{2}} + u^{-\frac{1}{2}}\right)du$$
$$= \frac{2}{3}\left[u^{\frac{3}{2}}\right]_1^4 + 2\left[u^{\frac{1}{2}}\right]_1^4 = \frac{2}{3}(8-1) + 2(2-1) = \frac{20}{3}$$

2 Let $u = 2 - x \Rightarrow du = -dx$

Limits: $x = 3 \Rightarrow u = -1, \ x = 4 \Rightarrow u = -2$

$$\Rightarrow \int_3^4 \left(\frac{x}{2-x}\right)^2 dx = \int_{-1}^{-2} \left(\frac{2-u}{u}\right)^2 (-du)$$
$$= \int_{-2}^{-1} \left(\frac{2-u}{u}\right)^2 du = \int_{-2}^{-1} \left(\frac{4}{u^2} - \frac{4}{u} + 1\right)du \quad .$$
$$= \left[-\frac{4}{u} - 4\ln|u| + u\right]_{-2}^{-1}$$
$$= (4-1) - (2 - 4\ln 2 - 2) = \ln 16 + 3$$

3 Let $u = 1 + \cos x \Rightarrow du = -\sin x dx$

Limits: $x = 0 \Rightarrow u = 2, \ x = \frac{\pi}{2} \Rightarrow u = 1$

$$\therefore \int_0^{\frac{\pi}{2}} \left(\frac{\sin x}{1+\cos x}\right)dx = -\int_2^1 \frac{1}{u}du = \int_1^2 \frac{1}{u}du$$
$$= \ln 2$$

4 Let $x = \frac{3}{2}\sin u \Rightarrow dx = \frac{3}{2}\cos u du$

Limits: $x = 0 \Rightarrow u = 0,\ x = \frac{3}{2} \Rightarrow u = \frac{\pi}{2}$

$$\int_0^{\frac{3}{2}} \frac{1}{\sqrt{9-4x^2}}dx = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{9-9\sin^2 u}}\left(\frac{3}{2}\cos u\right)du$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} du = \frac{\pi}{4}$$

5 Let $u = 2x-1 \Rightarrow du = 2dx$

Limits: $x = 1 \Rightarrow u = 1,\ x = 2 \Rightarrow u = 3$

$$\therefore \int_1^2 \frac{16x^4}{(2x-1)^2}dx = \frac{1}{2}\int_1^3 \frac{16\left(\frac{u+1}{2}\right)^4}{u^2}du$$

$$= \frac{1}{2}\int_1^3 \frac{(u+1)^4}{u^2}du = \frac{1}{2}\int_1^3\left(u^2+4u+6+\frac{4}{u}+\frac{1}{u^2}\right)du$$

$$= \frac{1}{2}\left[\frac{u^3}{3}+2u^2+6u+4\ln u-\frac{1}{u}\right]_1^3$$

$$= \frac{1}{2}\left(9+18+18+4\ln 3-\frac{1}{3}\right)-\frac{1}{2}\left(\frac{1}{3}+2+6-1\right)$$

$$= \frac{56}{3}+\ln 9$$

6 $\displaystyle\int_1^{\frac{3}{2}} \frac{(1-2x)\sqrt{1+x-x^2}}{1+x-x^2}dx = \int_1^{\frac{3}{2}} \frac{1-2x}{\sqrt{1+x-x^2}}dx$

Let $u = 1+x-x^2 \Rightarrow du = (1-2x)dx$

Limits: $x = 1 \Rightarrow u = 1, x = \frac{3}{2} \Rightarrow u = \frac{1}{4}$

$$\therefore \int_1^{\frac{3}{2}} \frac{(1-2x)\sqrt{1+x-x^2}}{1+x-x^2}dx = \int_1^{\frac{1}{4}} \frac{1}{u^{\frac{1}{2}}}du$$

$$= 2\left[u^{\frac{1}{2}}\right]_1^{\frac{1}{4}} = 2\left(\frac{1}{2}-1\right) = -1$$

7 Let $u = \sin x \Rightarrow du = \cos x dx$

Limits: $x = \frac{\pi}{4} \Rightarrow u = \frac{1}{\sqrt{2}}, x = \frac{\pi}{3} \Rightarrow u = \frac{\sqrt{3}}{2}$

$$\therefore \int_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} \frac{1}{u^3}du = \left[-\frac{1}{2u^2}\right]_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} = -\frac{1}{2}\left(\frac{4}{3}-2\right) = \frac{1}{3}$$

8 Let $u = \tan x \Rightarrow du = \sec^2 x dx$

Limits: $x = 0 \Rightarrow u = 0,\ x = \frac{\pi}{4} \Rightarrow u = 1$

$$\int_0^{\frac{\pi}{4}} \sec^2 x \tan^3 x dx = \int_0^1 u^3 du = \frac{1}{4}$$

9 $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{\sin x \cos x}{\cos^3 2x} dx = \frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{\sin 2x}{\cos^3 2x} dx$

Let $u = \cos 2x \Rightarrow du = -2\sin 2x dx$

Limits: $x = \frac{\pi}{6} \Rightarrow u = \frac{1}{2},\ x = \frac{\pi}{12} \Rightarrow u = \frac{\sqrt{3}}{2}$

$\therefore \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{\sin x \cos x}{\cos^3 2x} dx = -\frac{1}{4}\int_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} \frac{1}{u^3} du = \frac{1}{4}\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{u^3} du$.

$= \frac{1}{4}\left[-\frac{1}{2}u^{-2}\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = -\frac{1}{8}\left(\frac{4}{3} - 4\right) = \frac{1}{3}$

10 Let $u = 3^x \Rightarrow du = \ln 3 \cdot 3^x dx$

Limits: $x = 0 \Rightarrow u = 1,\ x = 2 \Rightarrow u = 9$

$\therefore \int_0^2 3^x \sqrt{3^x} dx = \frac{1}{\ln 3}\int_1^9 u^{\frac{1}{2}} du = \frac{2}{3\ln 3}\left[u^{\frac{3}{2}}\right]_1^9$

$= \frac{2}{3\ln 3}\left[27 - 1\right] = \frac{52}{3\ln 3}$

11 $\tan^3 x = \tan x \cdot \tan^2 x = \tan x\left(\sec^2 x - 1\right) = \sec^2 x \tan x - \tan x$

$\int_0^{\frac{\pi}{4}} \tan^3 x dx = \int_0^{\frac{\pi}{4}} \sec^2 x \tan x dx - \int_0^{\frac{\pi}{4}} \tan x dx$

Using question 8, and the fact that

$\int_0^{\frac{\pi}{4}} \tan x dx = -\left[\ln\left(\frac{1}{\sqrt{2}}\right)\right]$.

$\int_0^{\frac{\pi}{4}} \tan^3 x dx = \frac{1}{4} + \ln\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}\left(1 + 2\ln\left(\frac{1}{\sqrt{2}}\right)\right)$

$= \frac{1}{2}(1 - \ln 2)$ as required

12 Let $u = \cos kx \Rightarrow du = -k\sin kx\ dx$

Limits: $x = 0 \Rightarrow u = 1,\ x = \frac{\pi}{3} \Rightarrow u = \cos\left(\frac{k\pi}{3}\right)$

$\therefore \int_0^{\frac{\pi}{3}} \sin kx \cos^3 kx dx = -\frac{1}{k}\int_1^{\cos\left(\frac{k\pi}{3}\right)} u^3 du = -\frac{1}{k}\left(\frac{1}{4}\cos^4\left(\frac{k\pi}{3}\right) - \frac{1}{4}\right)$

$= \frac{1}{4k}\left(1 - \cos^4\left(\frac{k\pi}{3}\right)\right) = \frac{3}{16k}$

$\Rightarrow \cos^4\left(\frac{k\pi}{3}\right) = \frac{1}{4}$.

$\Rightarrow \cos\left(\frac{k\pi}{3}\right) = \frac{1}{\pm\sqrt{2}}$

$\Rightarrow \frac{k\pi}{3} = \frac{-1 + 8n}{4}\pi$ or $\frac{k\pi}{3} = \frac{1 + 8n}{4}\pi$ for $n \in$ □

$\Rightarrow k = \frac{-3}{4} + 6n$ or $k = \frac{3}{4} + 6n$ for $n \in$ □

13 $3x^2+12x+16=3\left(x^2+4x\right)+16$

$=3\left[\left(x+2\right)^2-4\right]+16=3\left(x+2\right)^2+4$

$$\int_{-2}^{\frac{2}{\sqrt{3}}-2}\frac{1}{3x^2+12x+16}dx=\int_{-2}^{\frac{2}{\sqrt{3}}-2}\frac{1}{4+3\left(x+2\right)^2}dx$$

Let $x=\frac{2\tan\theta}{\sqrt{3}}-2\Rightarrow dx=\frac{2}{\sqrt{3}}\sec^2\theta\,d\theta$

Limits: $x=\frac{2}{\sqrt{3}}-2\Rightarrow\theta=\frac{\pi}{4}$, $x=-2\Rightarrow\theta=0$

$$\therefore\int_{-2}^{\frac{2}{\sqrt{3}}-2}\frac{1}{3x^2+12x+16}dx=\frac{1}{4}\int_0^{\frac{\pi}{4}}\frac{1}{1+\frac{3}{4}\left(\frac{2}{\sqrt{3}}\tan\theta\right)^2}\left(\frac{2}{\sqrt{3}}\sec^2\theta\right)d\theta$$

$$=\frac{1}{2\sqrt{3}}\int_0^{\frac{\pi}{4}}\frac{\sec^2\theta}{1+\tan^2\theta}d\theta=\frac{1}{2\sqrt{3}}\int_0^{\frac{\pi}{4}}d\theta=\frac{\pi}{8\sqrt{3}}\quad\text{as required}$$

14 $e^x+e^{-x}=e^{-x}\left(e^{2x}+1\right)=\frac{e^{2x}+1}{e^x}$

$$\therefore\int_0^{-\ln\sqrt{3}}\frac{1}{e^x+e^{-x}}dx=\int_0^{-\ln\sqrt{3}}\frac{e^x}{e^{2x}+1}dx$$

Let $e^x=\tan\theta\Rightarrow e^xdx=\sec^2\theta d\theta=\left(e^{2x}+1\right)d\theta$

Limits: $x=-\ln\sqrt{3}\Rightarrow\theta=\frac{\pi}{6}$, $x=0\Rightarrow\theta=\frac{\pi}{4}$

$$\therefore\int_0^{-\ln\sqrt{3}}\frac{1}{e^x+e^{-x}}dx=\int_{\frac{\pi}{4}}^{\frac{\pi}{6}}d\theta=\frac{\pi}{6}-\frac{\pi}{4}=-\frac{\pi}{12}$$

15 a There are many ways to do this, the easiest being:

$$\tan x=\frac{2\tan\frac{x}{2}}{1-\tan^2\frac{1}{2}}=\frac{2t}{1-t^2}$$

Construct RA triangle with opposite side of length $2t$ and adjacent of length $1-t^2$

Then the hypotenuse is of length

$$\sqrt{\left(1-t^2\right)^2+\left(2t\right)^2}=\sqrt{1+2t^2+t^4}=\sqrt{\left(1+t^2\right)^2}=1+t^2\Rightarrow\sin x=\frac{2t}{1+t^2}\quad\text{as required}$$

b Differentiating $\sin x=\frac{2t}{1+t^2}$ implicitly gives

$$\frac{dx}{dt}\cos x=\frac{2\left(1+t^2\right)-\left(2t\right)\left(2t\right)}{\left(1+t^2\right)^2}=\frac{2-2t^2}{\left(1+t^2\right)^2}=\frac{2\left(1-t^2\right)}{\left(1+t^2\right)^2}$$

$$\cos x=\frac{1-t^2}{1+t^2}$$

$$\Rightarrow\frac{dx}{dt}=\frac{1+t^2}{1-t^2}\cdot\frac{2\left(1-t^2\right)}{\left(1+t^2\right)^2}=\frac{2}{1+t^2}\quad\text{as required}$$

c Let $t = \tan\left(\frac{x}{2}\right) \Rightarrow dx = \frac{2dt}{1+t^2}$

Limits: $x = \frac{\pi}{2} \Rightarrow t = 1,\ x = 0 \Rightarrow t = 0$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}dx = \int_0^1 \frac{1}{1+\frac{2t}{1+t^2}}\left(\frac{2}{1+t^2}\right)dt$$

$$= \int_0^1 \frac{2}{1+t^2+2t}dt = \int_0^1 \frac{2}{(1+t)^2}dt$$

$$= -\left[\frac{2}{1+t}\right]_0^1 = -(1-2) = 1 \quad \text{as required}$$

Exercise 7J

1 $\int 2xe^x dx = 2\int xe^x dx$

Let $u = x$ and $\frac{dv}{dx} = e^x$

so $\frac{du}{dx} = 1$ and $v = e^x$

$\therefore \int 2xe^x dx = 2\left(xe^x - \int e^x dx\right) = 2e^x(x-1) + C$

2 Let $u = 3x$ and $\frac{dv}{dx} = \sin x$

so $\frac{du}{dx} = 3$ and $v = -\cos x$

$\therefore \int 3x\sin x dx = -3x\cos x + 3\int \cos x dx$

$= -3x\cos x + 3\sin x + C$

$= 3(\sin x - x\cos x) + C$

3 Let $u = 1-2x$ and $\frac{dv}{dx} = e^x$

so $\frac{du}{dx} = -2$ and $v = e^x$

$\therefore \int (1-2x)e^x dx = e^x(1-2x) + 2\int e^x dx$

$= e^x(3-2x) + C$

4 Let $u = 2-x$ and $\frac{dv}{dx} = \sin(2-x)$

so $\frac{du}{dx} = -1$ and $v = \cos(2-x)$

$\therefore \int (2-x)\sin(2-x)dx = (2-x)\cos(2-x) + \int \cos(2-x)dx$

$= (2-x)\cos(2-x) - \sin(2-x) + C$

5 Let $u = \frac{1+2x}{3}$ and $\frac{dv}{dx} = \sec^2 \frac{x}{2}$

so $\frac{du}{dx} = \frac{2}{3}$ and $v = 2\tan\frac{x}{2}$

$$\therefore \int\left(\frac{1+2x}{3}\right)\sec^2\frac{x}{2}\,dx = \frac{2}{3}(1+2x)\tan\frac{x}{2} - \frac{4}{3}\int\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}dx$$

$$= \frac{2}{3}(1+2x)\tan\frac{x}{2} + \frac{8}{3}\ln\left(\cos\frac{x}{2}\right) + C$$

6 Let $u = x$ and $\frac{dv}{dx} = 2^{x+1}$

so $\frac{du}{dx} = 1$ and $v = \frac{1}{\ln 2}2^{x+1}$

$$\therefore \int x2^{x+1}dx = \frac{x2^{x+1}}{\ln 2} - \frac{1}{\ln 2}\int 2^{x+1}dx = \frac{x2^{x+1}}{\ln 2} - \frac{1}{(\ln 2)^2}2^{x+1} + C$$

$$= \frac{2^{x+1}}{\ln 2}\left(x - \frac{1}{\ln 2}\right) + C$$

7 Let $u = x$ and $\frac{dv}{dx} = 3^{-x}$

so $\frac{du}{dx} = 1$ and $v = -\frac{1}{\ln 3}3^{-x}$

$$\therefore \int\frac{x}{3^x}dx = -\frac{x3^{-x}}{\ln 3} + \frac{1}{\ln 3}\int 3^{-x}dx = -\frac{x3^{-x}}{\ln 3} - \frac{1}{(\ln 3)^2}3^{-x} + C$$

$$= -\frac{3^{-x}}{\ln 3}\left(x + \frac{1}{\ln 3}\right) + C$$

8 Let $u = \ln x$ and $\frac{dv}{dx} = x^3$

so $\frac{du}{dx} = \frac{1}{x}$ and $v = \frac{x^4}{4}$

$$\therefore \int x^3\ln x\,dx = \frac{1}{4}x^4\ln x - \frac{1}{4}\int x^3dx$$

$$= \frac{1}{4}x^4\ln x - \frac{1}{16}x^4 + C$$

$$= \frac{x^4}{16}(4\ln x - 1) + C$$

9 Let $u = \ln\frac{x}{3}$ and $\frac{dv}{dx} = 2 - 5x$

so $\frac{du}{dx} = \frac{1}{x}$ and $v = 2x - \frac{5}{2}x^2$

$$\therefore \int(2-5x)\ln\frac{x}{3}dx = \left(2x - \frac{5}{2}x^2\right)\ln\frac{x}{3} - \int\left(2 - \frac{5}{2}x\right)dx$$

$$= \left(2x - \frac{5}{2}x^2\right)\ln\frac{x}{3} - 2x + \frac{5}{4}x^2 + C$$

10 Let $u = \arcsin x$ and $\frac{dv}{dx} = 1$

so $\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$ and $v = x$

$\therefore \int \arcsin x dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx$

Using substitution to evaluate this integral:

Let $u = 1 - x^2 \Rightarrow du = -2x\,dx$, .

$\int \frac{x}{\sqrt{1-x^2}} dx = -\frac{1}{2}\int \frac{du}{\sqrt{u}}$

$= -\frac{1}{2}\int u^{-\frac{1}{2}} du = -u^{\frac{1}{2}} = -\sqrt{1-x^2}$

$\int \arcsin x\, dx = x \arcsin x + \sqrt{1-x^2} + C$

11 Let $u = \ln 4x$ and $\frac{dv}{dx} = 1 + 3x - x^2$

so $\frac{du}{dx} = \frac{1}{x}$ and $v = x + \frac{3}{2}x^2 - \frac{1}{3}x^3$

$\therefore \int (1 + 3x - x^2)\ln 4x\ dx = \left(x + \frac{3}{2}x^2 - \frac{1}{3}x^3\right)\ln 4x - \int \left(1 + \frac{3}{2}x - \frac{1}{3}x^2\right)dx$

$= \left(x + \frac{3}{2}x^2 - \frac{1}{3}x^3\right)\ln 4x - x - \frac{3}{4}x^2 + \frac{1}{9}x^3 + C$

12 $\int \log_a x dx = \frac{1}{\ln a}\int 1 \cdot \ln x dx$

Let $u = \ln x$ and $\frac{dv}{dx} = 1$

so $\frac{du}{dx} = \frac{1}{x}$ and $v = x$

$\therefore \int \log_a x dx = \frac{1}{\ln a}\left(x \ln x - \int dx\right)$

$= x\frac{\ln x}{\ln a} - \frac{x}{\ln a} + C$

$= \frac{x}{\ln a}(\ln x - 1) + C$

13 Let $u = \arccos x$ and $\frac{dv}{dx} = x$, then $\frac{du}{dx} = -\frac{1}{\sqrt{1-x^2}}$ and $v = \frac{x^2}{2}$

$\Rightarrow \int x \arccos x\, dx = \frac{x^2 \arccos x}{2} - \frac{1}{2}\int \frac{(1-x^2)-1}{\sqrt{1-x^2}} dx$

$= \frac{x^2 \arccos x}{2} - \frac{1}{2}\left[\int \sqrt{1-x^2}\, dx - \int \frac{1}{\sqrt{1-x^2}} dx\right]$

$= \frac{x^2 \arccos x}{2} + \frac{1}{2}\arcsin x - \frac{1}{2}\int \sqrt{1-x^2}\, dx$

Use substitution to evaluate this integral:

Let $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

Then $\int \sqrt{1-x^2}\, dx = \int \sqrt{1-\sin^2\theta}\, \cos\theta\, d\theta$

$= \int \cos^2\theta\, d\theta$ (use the identity $\cos 2\theta \equiv 2\cos^2\theta - 1$)

$= \frac{1}{2}\int (\cos 2\theta + 1)\, d\theta$

$= \frac{1}{4}\sin 2\theta + \theta = \frac{1}{2}\sin\theta\cos\theta + \theta$ (Substitute back x for θ)

$= \frac{1}{2}x\sqrt{1-x^2} + \arcsin x$

Hence

$\int x \arccos x\, dx = \frac{x^2 \arccos x}{2} + \frac{1}{4}\arcsin x - \frac{1}{4}x\sqrt{1-x^2} + c$

14 Let $u = \arctan x$ and $\frac{dv}{dx} = 4x$

so $\frac{du}{dx} = \frac{1}{1+x^2}$ and $v = 2x^2$

$\therefore \int 4x \arctan x dx = 2x^2 \arctan x - 2\int \frac{x^2}{1+x^2} dx$

$= 2x^2 \arctan x - 2\int \frac{1+x^2-1}{1+x^2} dx$

$= 2x^2 \arctan x - 2\int \left(1 - \frac{1}{1+x^2}\right) dx$

$= 2x^2 \arctan x - 2(x - \arctan x) + C$

$= 2(x^2+1)\arctan x - 2x + C$

15 $I = \int x^2 \arccos x$

Let $u = \arccos x$ $\quad dv = x^2$

$du = -\frac{1}{\sqrt{1-x^2}}$ $\quad v = \frac{x^3}{3}$

$I = \frac{x^3}{3}\arccos x + \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}} dx$

Use substitution to evaluate this integral

Let $u = 1 - x^2 \Rightarrow du = -2x\, dx \Rightarrow x\, dx = -\frac{1}{2}du$

$I = \frac{x^3}{3}\arccos x - \frac{1}{6}\int \frac{1-u}{\sqrt{u}}\, du$

$= \frac{x^3}{3}\arccos x - \frac{1}{6}\int \left(u^{-\frac{1}{2}} - u^{\frac{1}{2}}\right) du$

$= \frac{x^3}{3}\arccos x - \frac{1}{6}\left(2u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right) + c$

$= \frac{x^3}{3}\arccos x - \frac{u^{\frac{1}{2}}}{3}\left(1 - \frac{u}{3}\right) + c$ (Substitute back x for u and simplify)

$= \frac{x^3}{3}\arccos x - \frac{\sqrt{1-x^2}}{3}\left(1 - \frac{(1-x^2)}{3}\right) + c$

$= \frac{x^3}{3}\arccos x - \frac{\sqrt{1-x^2}}{9}(2+x^2) + c$

16 Let $u = \arctan x$ and $\frac{dv}{dx} = 1$

so $\frac{du}{dx} = \frac{1}{1+x^2}$ and $v = x$

$$\therefore \int_0^{\sqrt{3}} \arctan x dx = \left[x \arctan x\right]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{x}{1+x^2} dx$$

$$= \sqrt{3}\arctan\sqrt{3} - \frac{1}{2}\left[\ln\left(1+x^2\right)\right]_0^{\sqrt{3}}$$

$$= \sqrt{3}\arctan\sqrt{3} - \frac{1}{2}\ln 4$$

$$= \sqrt{3}\arctan\sqrt{3} - \ln 2$$

17 Let $u = \ln x$ and $\frac{dv}{dx} = x^{-5}$

so $\frac{du}{dx} = \frac{1}{x}$ and $v = -\frac{1}{4}x^{-4}$

$$\therefore \int_1^3 x^{-5}\ln x dx = -\frac{1}{4}\left[x^{-4}\ln x\right]_1^3 + \frac{1}{4}\int_1^3 x^{-5} dx$$

$$= -\frac{\ln 3}{324} - \frac{1}{16}\left[x^{-4}\right]_1^3$$

$$= -\frac{\ln 3}{324} - \frac{1}{16}\left(\frac{1}{81} - 1\right) = -\frac{\ln 3}{324} + \frac{5}{81}$$

$$= \frac{1}{324}(20 - \ln 3)$$

18 Let $u = x$ and $\frac{dv}{dx} = \sec^2 x$

so $\frac{du}{dx} = 1$ and $v = \tan x$

$$\int_0^{\frac{\pi}{4}} \left(\frac{x}{\cos^2 x}\right) dx = \left[x\tan x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x dx$$

$$= \frac{\pi}{4} + \left[\ln(\cos x)\right]_0^{\frac{\pi}{4}} = \frac{\pi}{4} - \ln\left(\sqrt{2}\right)$$

19 Let $u = \ln(3x)$ and $\frac{dv}{dx} = 1 - 2x + x^2$

so $\frac{du}{dx} = \frac{1}{x}$ and $v = x - x^2 + \frac{x^3}{3}$

$$\therefore \int_1^e \left(1 - 2x + x^2\right)\ln(3x) dx = \left[\left(x - x^2 + \frac{x^3}{3}\right)\ln(3x)\right]_1^e - \int_1^e \left(1 - x + \frac{x^2}{3}\right) dx$$

$$= \left(e - e^2 + \frac{e^3}{3}\right)\ln(3e) - \frac{1}{3}\ln 3 - \left[x - \frac{x^2}{2} + \frac{x^3}{9}\right]_1^e$$

$$= \left(e - e^2 + \frac{e^3}{3}\right)\ln(3e) - \frac{1}{3}\ln 3 - \left(e - \frac{e^2}{2} + \frac{e^3}{9} - \frac{11}{18}\right)$$

$$= \left(e - e^2 + \frac{e^3}{3} - \frac{1}{3}\right)\ln(3) - \frac{e^2}{2} + \frac{2e^3}{9} + \frac{11}{18}$$

20 Let $u = x$ and $\frac{dv}{dx} = \sin x$

so $\frac{du}{dx} = 1$ and $v = -\cos x$

$\therefore \int_0^{\frac{\pi}{4}} x \sin x dx = \left[-x \cos x\right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \cos x dx$

$= -\frac{\pi}{4\sqrt{2}} + \left[\sin x\right]_0^{\frac{\pi}{4}}$

$= -\frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{\sqrt{2}(4-\pi)}{8}$

Exercise 7K

1 Integrating by parts

Let $u = 2x^2 \quad dv = e^{2x}$

$du = 4x \quad v = \frac{1}{2}e^{2x}$

$\int 2x^2 e^{2x} dx = x^2 e^{2x} - 2\int x e^{2x} dx$

Integrating by parts again

Let $u = x \quad dv = e^{2x}$

$du = dx \quad v = \frac{1}{2}e^{2x}$

$= x^2 e^{2x} - 2\left(\frac{1}{2}x e^{2x} - \frac{1}{2}\int e^{2x} dx\right)$

$= x^2 e^{2x} - x e^{2x} + \frac{1}{2}e^{2x} + C$

$= \left(x^2 - x + \frac{1}{2}\right)e^{2x} + C$

$= \frac{e^{2x}}{2}\left(2x^2 - 2x + 1\right) + C$

2 Integrating by parts

Let $u = x^3 \quad dv = \sin x$

$du = 3x^2 \quad v = -\cos x$

$\int x^3 \sin x dx = -x^3 \cos x + 3\int x^2 \cos x dx$

Integrating by parts again

Let $u = x^2 \quad dv = \cos x$

$du = 2x \quad v = \sin x$

$= -x^3 \cos x + 3\left(x^2 \sin x - \int 2x \sin x dx\right)$

$= -x^3 \cos x + 3x^2 \sin x - 6\int x \sin x dx$

Integrating by parts again

Let $u = x \quad dv = \sin x$

$du = dx \quad v = -\cos x$

$= -x^3 \cos x + 3x^2 \sin x - 6\left(-x \cos x + \int \cos x dx\right)$

$= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$

$= 3\left(x^2 - 2\right)\sin x + x\left(6 - x^2\right)\cos x + C$

3 Integrating by parts

Let $u = x - x^2 \quad dv = \cos x$
$du = 1 - 2x \quad v = \sin x$
$\int (x - x^2) \cos x dx$
$= (x - x^2) \sin x - \int (1 - 2x) \sin x dx$
Integrating by parts again
Let $u = 1 - 2x \quad dv = \sin x$
$du = -2 \quad v = -\cos x$
$= (x - x^2) \sin x - \left[(2x - 1) \cos x - 2\int \cos x dx\right]$
$= (x - x^2) \sin x + (1 - 2x) \cos x + 2 \sin x + C$
$= (2 + x - x^2) \sin x + (1 - 2x) \cos x + C$

4 Integrate by parts

Let $u = x^2 \quad dv = \sin\left(\frac{x}{4}\right)$

$du = 2x \quad v = -4\cos\left(\frac{x}{4}\right)$

$\int x^2 \sin\left(\frac{x}{4}\right) dx = -4x^2 \cos\left(\frac{x}{4}\right) + 8\int x \cos\left(\frac{x}{4}\right) dx$
Integrate by parts again

Let $u = x \quad dv = \cos\left(\frac{x}{4}\right)$

$du = dx \quad v = 4\sin\left(\frac{x}{4}\right)$

$= -4x^2 \cos\left(\frac{x}{4}\right) + 32x \sin\left(\frac{x}{4}\right) - 32\int \sin\left(\frac{x}{4}\right) dx$

$= -4x^2 \cos\left(\frac{x}{4}\right) + 32x \sin\left(\frac{x}{4}\right) + 128\cos\left(\frac{x}{4}\right) + C$

$= -4(x^2 - 32)\cos\left(\frac{x}{4}\right) + 32x \sin\left(\frac{x}{4}\right) + C$

5 Integrating by parts

Let $u = x^3 \quad dv = e^{\frac{x}{3}}$

$du = 3x^2 \quad v = 3e^{\frac{x}{3}}$

$\int x^3 e^{\frac{x}{3}} dx = 3x^3 e^{\frac{x}{3}} - 9\int x^2 e^{\frac{x}{3}} dx$
Integrate by parts again

Let $u = x^2 \quad dv = e^{\frac{x}{3}}$

$du = 2x \quad v = 3e^{\frac{x}{3}}$

$= 3x^3 e^{\frac{x}{3}} - 9\left(3x^2 e^{\frac{x}{3}} - 6\int x e^{\frac{x}{3}} dx\right)$

$= 3x^3 e^{\frac{x}{3}} - 27x^2 e^{\frac{x}{3}} + 54\int x e^{\frac{x}{3}} dx$

Integrate by parts again

Let $u = x \qquad dv = e^{\frac{x}{3}}$

$du = dx \qquad v = 3e^{\frac{x}{3}}$

$= 3x^3e^{\frac{x}{3}} - 27x^2e^{\frac{x}{3}} + 54\left(3xe^{\frac{x}{3}} - 3\int e^{\frac{x}{3}}dx\right)$

$= 3x^3e^{\frac{x}{3}} - 27x^2e^{\frac{x}{3}} + 162xe^{\frac{x}{3}} - 486e^{\frac{x}{3}} + C$

$= 3e^{\frac{x}{3}}\left(x^3 - 9x^2 + 54x - 162\right) + C$

6 Integrating by parts

Let $u = x^2 \qquad dv = e^x$

$du = 2x \qquad v = e^x$

$\int_0^2 x^2e^xdx = \left[x^2e^x\right]_0^2 - 2\int_0^2 xe^xdx$

Integrating by parts again

Let $u = x \qquad dv = e^x$

$du = dx \qquad v = e^x$

$= \left[x^2e^x\right]_0^2 - 2\left(\left[xe^x\right]_0^2 - \int_0^2 e^xdx\right)$

$= \left[x^2e^x\right]_0^2 - 2\left(\left[xe^x\right]_0^2 - \left[e^x\right]_0^2\right)$

$= \left[4e^2 - 0\right] - 2\left[2e^2 - 0\right] + 2\left[e^2 - 1\right]$

$= 4e^2 - 4e^2 + 2\left(e^2 - 1\right)$

$= 2\left(e^2 - 1\right)$

7 Integrating by parts

Let $u = x^2 \qquad dv = \sin x$

$du = 2x \qquad v = -\cos x$

$\int_0^{\frac{\pi}{2}} x^2 \sin x dx = \left[-x^2 \cos x\right]_0^{\frac{\pi}{2}} + 2\int_0^{\frac{\pi}{2}} x \cos x dx$

Integrating by parts again

Let $u = x \qquad dv = \cos x$

$du = dx \qquad v = \sin x$

$= \left[-x^2 \cos x\right]_0^{\frac{\pi}{2}} + 2\left(\left[x \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x dx\right)$

$= \left[-x^2 \cos x\right]_0^{\frac{\pi}{2}} + 2\left[x \sin x\right]_0^{\frac{\pi}{2}} - 2\left[-\cos x\right]_0^{\frac{\pi}{2}}$

$= \left[0 - 0\right] + 2\left[\frac{\pi}{2} - 0\right] - 2\left[-0 - (-1)\right]$

$= \pi - 2$

8 Integrating by parts

Let $u = \left(1 + x^2\right) \qquad dv = \cos x$

$du = 2x \qquad v = \sin x$

$\int_0^{\frac{\pi}{2}} \left(1 + x^2\right)\cos x dx = \left[\left(1 + x^2\right)\sin x\right]_0^{\frac{\pi}{2}} - 2\int_0^{\frac{\pi}{2}} x \sin x dx$

Integrating by parts again

Let $u = x \quad dv = \sin x$

$du = dx \quad v = -\cos x$

$$= \left[\left(1+x^2\right)\sin x\right]_0^{\frac{\pi}{2}} - 2\left(-\left[x\cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\cos x\,dx\right)$$

$$= \left[\left(1+x^2\right)\sin x\right]_0^{\frac{\pi}{2}} + 2\left[x\cos x\right]_0^{\frac{\pi}{2}} - 2\left[\sin x\right]_0^{\frac{\pi}{2}}$$

$$= \left[\left(1+\frac{\pi^2}{4}\right)-0\right] + 2\left[0-0\right] - 2\left[1-0\right]$$

$$= \left(1+\frac{\pi^2}{4}\right) - 2 = \frac{\pi^2-4}{4}$$

9 Integrating by parts

Let $u = x^2 \quad dv = e^{3x}$

$du = 2x \quad v = \frac{e^{3x}}{3}$

$$\frac{1}{3}\int_0^1 x^2e^{3x}\,dx = \frac{1}{3}\left\{\left[\frac{1}{3}x^2e^{3x}\right]_0^1 - \frac{2}{3}\int_0^1 xe^{3x}\,dx\right\}$$

Integrating by parts again

Let $u = x \quad dv = e^{3x}$

$du = dx \quad v = \frac{e^{3x}}{3}$

$$= \frac{1}{3}\left[\frac{1}{3}x^2e^{3x}\right]_0^1 - \frac{2}{9}\left\{\left[\frac{1}{3}xe^{3x}\right]_0^1 - \frac{1}{3}\int_0^1 e^{3x}dx\right\}$$

$$= \frac{1}{3}\left[\frac{1}{3}x^2e^{3x}\right]_0^1 - \frac{2}{9}\left[\frac{1}{3}xe^{3x}\right]_0^1 - \frac{2}{27}\left[\frac{1}{3}e^{3x}\right]_0^1$$

$$= \frac{1}{3}\left(\frac{e^3}{3}-0\right) - \frac{2}{9}\left(\frac{e^3}{3}-0\right) - \frac{2}{27}\left(\frac{e^3}{3}-\frac{1}{3}\right)$$

$$= \frac{1}{9}e^3 - \frac{2}{27}e^3 + \frac{2}{81}\left(e^3-1\right)$$

$$= \frac{5e^3-2}{81}$$

10 Integrating by parts

Let $u = x^2 \quad dv = e^{-2x}$

$du = 2x \quad v = -\frac{e^{-2x}}{2}$

$$2\int_0^1 x^2e^{-2x}dx = 2\left\{\left[-\frac{1}{2}x^2e^{-2x}\right]_0^1 + \int_0^1 xe^{-2x}dx\right\}$$

Simplifying and integrating by parts again

Let $u = x \quad dv = e^{-2x}$

$du = dx \quad v = -\dfrac{e^{-2x}}{2}$

$$= 2\left[-\frac{1}{2}x^2e^{-2x}\right]_0^1 + 2\left\{\left[-\frac{1}{2}xe^{-2x}\right]_0^1 + \frac{1}{2}\int_0^1 e^{-2x}dx\right\}$$

$$= 2\left[-\frac{1}{2}x^2e^{-2x}\right]_0^1 + 2\left[-\frac{1}{2}xe^{-2x}\right]_0^1 + \left[-\frac{1}{2}e^{-2x}\right]_0^1$$

$$= 2\left(-\frac{1}{2}e^{-2} - 0\right) + 2\left(-\frac{1}{2}e^{-2} - 0\right) + \left(-\frac{1}{2}e^{-2} - \left(-\frac{1}{2}\right)\right)$$

$$= -2e^{-2} - \frac{1}{2}e^{-2} + \frac{1}{2}$$

$$= \frac{1 - 5e^{-2}}{2}$$

$$= \frac{1}{2}\left(1 - \frac{5}{e^2}\right)$$

Exercise 7L

1 Let $I = \int \tan x \sec^2 x dx$

Integrating by parts,

Let $u = \tan x \quad dv = \sec^2 x$

$du = \sec^2 x \quad v = \tan x$

$I = \int \tan x \sec^2 x dx = \tan^2 x - \int \tan x \sec^2 x dx$

$\Rightarrow I = \tan^2 x - I$

$\Rightarrow I = \dfrac{\tan^2 x}{2}$

$\therefore I = \dfrac{\tan^2 x}{2} + C$ (redefine I to include constant)

2 Let $I = \int \sin x \cos x dx$

Integrating by parts,

Let $u = \sin x \quad dv = \cos x$

$du = \cos x \quad v = \sin x$

$I = \sin^2 x - \int \sin x \cos x \, dx$

$\Rightarrow I = \sin^2 x - I$

$\Rightarrow I = \dfrac{1}{2}\sin^2 x$

$\therefore I = \dfrac{1}{2}\sin^2 x + C$ (redefine I to include constant)

3 Let $I = \int \sin 2x \cos 3x dx$

Integrating by parts,

Let $u = \sin 2x \quad dv = \cos 3x$

$du = 2\cos 2x \quad v = \frac{1}{3}\sin 3x$

$I = \frac{1}{3}\sin 2x \sin 3x - \frac{2}{3}\int \sin 3x \cos 2x dx$

Integrating by parts again,

Let $u = \cos 2x \quad dv = \sin 3x$

$du = -2\sin 2x \quad v = -\frac{1}{3}\cos 3x$

$I = \frac{1}{3}\sin 2x \sin 3x - \frac{2}{3}\left(-\frac{1}{3}\cos 2x \cos 3x - \frac{2}{3}\int \sin 2x \cos 3x\, dx\right)$

$\Rightarrow I = \frac{1}{3}\sin 2x \sin 3x + \frac{2}{9}\cos 2x \cos 3x + \frac{4}{9}I$

$\therefore \frac{5}{9}I = \frac{1}{3}\left(\sin 2x \sin 3x + \frac{2}{3}\cos 2x \cos 3x\right)$

$\therefore I = \frac{1}{5}(3\sin 2x \sin 3x + 2\cos 2x \cos 3x) + C$ (redefine I to include constant)

4 Let $I = \int e^{3x} \cos 2x dx$

Integrating by parts,

Let $u = e^{3x} \quad dv = \cos 2x$

$du = 3e^{3x} \quad v = \frac{1}{2}\sin 2x$

$I = \frac{1}{2}e^{3x}\sin 2x - \frac{3}{2}\int e^{3x}\sin 2x dx$

Integrating by parts again

Let $u = e^{3x} \quad dv = \sin 2x$

$du = 3e^{3x} \quad v = -\frac{1}{2}\cos 2x$

$I = \frac{1}{2}e^{3x}\sin 2x - \frac{3}{2}\left[-\frac{1}{2}e^{3x}\cos 2x + \frac{3}{2}\int e^{3x}\cos 2x\, dx\right]$

$\Rightarrow I = \frac{1}{2}e^{3x}\sin 2x + \frac{3}{4}e^{3x}\cos 2x - \frac{9}{4}I$

$\Rightarrow \frac{13}{4}I = \frac{1}{2}e^{3x}\sin 2x + \frac{3}{4}e^{3x}\cos 2x$

$\Rightarrow \frac{13}{4}I = \frac{e^{3x}}{4}(2\sin 2x + 3\cos 2x)$

$\therefore I = \frac{e^{3x}}{13}(2\sin 2x + 3\cos 2x) + C$ (redefine I to include constant)

5 Let $I = \int \sin^2 x dx = \int \sin x \sin x dx$

Integrating by parts

Let $u = \sin x \quad dv = \sin x$

$du = \cos x \quad v = -\cos x$

$I = -\sin x \cos x + \int \cos^2 x dx$

$\Rightarrow I = -\sin x \cos x + \int (1 - \sin^2 x) dx$

$\Rightarrow I = -\sin x \cos x + \int dx - I$

$\Rightarrow 2I = -\sin x \cos x + x$

$\therefore I = \frac{1}{2}(x - \sin x \cos x) + C$ (redefine I to include constant)

Chapter review

1 a $(2^2)^{x-1} - 3(2^x) + 8 = 0$

$\Rightarrow \frac{1}{4}(2^x)^2 - 3(2^x) + 8 = 0$

$\Rightarrow (2^x)^2 - 12(2^x) + 32 = 0$

$\Rightarrow (2^x - 8)(2^x - 4) = 0$

$2^x - 8 = 0 \Rightarrow x = 3$

$2^x = 4 \Rightarrow x = 2$

So $x = 2$ or $x = 3$

b $2(5^x) - 9 = \frac{1}{5^{x-1}}$

Multiply throughout by 5^x

$\Rightarrow 2(5^x)^2 - 9(5^x) - 5 = 0$

$\Rightarrow (2 \times 5^x + 1)(5^x - 5) = 0$

$2 \times 5^x + 1 = 0 \Rightarrow$ has no solutions since $5^x > 0$ for all $x \in \mathbb{R}$

$5^x - 5 = 0 \Rightarrow x = 1$

2 a $\log_2 x + \log_2 3 - \log_2 5 = \log_2 6$

$\Rightarrow \log_2\left(\frac{3x}{5}\right) = \log_2 6$

$\therefore \frac{3x}{5} = 6 \Rightarrow x = 10$

b $\log_9 x - \log_9 7 = \log_9\left(\frac{x}{7}\right) = \frac{3}{2}$

$\therefore \frac{x}{7} = 9^{\frac{3}{2}} = 27 \Rightarrow x = 189$

3 a $\log_2 x - \frac{3}{\log_x 2} = \log_2 x - 3\log_2 x = -2\log_2 x = 4$

$\therefore \log_2 x = -2 \Rightarrow x = 2^{-2} = \frac{1}{4}$

b $\log_7 x - 4\log_x 7 = 0$

$\Rightarrow \log_7 x - \frac{4}{\log_7 x} = 0$

$$\therefore (\log_7 x)^2 = 4 \Rightarrow \log_7 x = \pm 2$$
$$\log_7 x = 2 \Rightarrow x = 7^2 = 49$$
$$\log_7 x = -2 \Rightarrow x = \frac{1}{49}$$

4 $\frac{dy}{dx} = 6e^{3x} + 21e^{-3x}$

$$\frac{d^2y}{dx^2} = 18e^{3x} - 63e^{-3x} = 9\left(2e^{3x} - 7e^{-3x}\right) = 9y$$

5 $\ln x + \ln x^2 + \ldots + \ln x^m = \ln x + 2\ln x + \ldots + m\ln x$

$$= (1+2+\ldots.+m)\ln x = \frac{1}{2}m(m+1)\ln x = 2m(m+1)$$
$$\Rightarrow \ln x = 4 \Rightarrow x = e^4$$

6 $\ln x - \frac{1}{\ln x} = 4 \Rightarrow (\ln x)^2 - 4(\ln x) - 1 = 0$

Let $\ln x = y$
Then
$y^2 - 4y - 1 = 0$
$y = \frac{4 \pm \sqrt{16+4}}{2} = 2 \pm \sqrt{5}$

7 $f'(x) = \frac{\left(1+2e^{2x}\right)\cdot 3e^x - 3e^x\left(4e^{2x}\right)}{\left(1+2e^{2x}\right)^2} = \frac{3e^x - 6e^{3x}}{\left(1+2e^{2x}\right)^2}$

$f'(x) = 0$
$\Rightarrow 3e^x - 6e^{3x} = 3e^x\left(1-2e^{2x}\right) = 0$
$\therefore e^{2x} = \frac{1}{2} \Rightarrow x = \frac{1}{2}\ln\frac{1}{2}$

Justification of maximum either by demonstrating $f''\left(\frac{1}{2}\ln\frac{1}{2}\right) < 0$

or alternatively by arguing that $\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = 0,$

and $f\left(\frac{1}{2}\ln\frac{1}{2}\right) > 0$ so since $f(x)$ is continuous, the single turning point must be a maximum

Therefore the coordinates of the maximum point are

$$\left(\frac{1}{2}\ln\frac{1}{2}, f\left(\frac{1}{2}\ln\frac{1}{2}\right)\right) = \left(\frac{1}{2}\ln\frac{1}{2}, \frac{3\sqrt{2}}{4}\right) = \left(-\ln\sqrt{2}, \frac{3\sqrt{2}}{4}\right)$$

8 For $x > 1$, $y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x} = \frac{\ln x}{x\ln x} = \frac{\ln x}{x|\ln x|}$

For $0 < x < 1$, $y = |\ln x| = -\ln x$
$\Rightarrow \frac{dy}{dx} = -\frac{1}{x} = -\frac{\ln x}{x\ln x} = \frac{\ln x}{x(-\ln x)} = \frac{\ln x}{x|\ln x|}$

So $\frac{dy}{dx} = \frac{\ln x}{x|\ln x|}$

9 **a** $f'(x)=3\cdot\frac{1}{x}(\ln x)^2=\frac{3}{x}(\ln x)^2$

$f''(x)=-\frac{3}{x^2}(\ln x)^2+\frac{3}{x}\cdot\frac{2}{x}\ln x=-\frac{3}{x^2}(\ln x)^2+\frac{6}{x^2}\ln x$

$\therefore f''(x)=\frac{3}{x^2}(\ln x)(-\ln x+2)=0$

So $\ln x=0$ or $\ln x=2$

$\ln x=0\Rightarrow x=1$

$\ln x=2\Rightarrow x=e^2>2^2=4$ which is outside the domain

Therefore the only point of inflection is $(1,0)$

b $f'(x)=\frac{3}{x}(\ln x)^2$ $\quad f'(x)=\frac{3}{x}(\ln x)^2\Rightarrow f'(e)=\frac{3}{e}$

Tangent:

$y-1=\frac{3}{e}(x-e)\Rightarrow y=\frac{3}{e}x-2$

Normal:

$y-1=\frac{-e}{3}(x-e)\Rightarrow y=1+\frac{e^2}{3}-\frac{e}{3}x$

c Tangent intersects the $y-$axis at $(0,-2)$

Normal intersects the $y-$axis at $\left(0,1+\frac{e^2}{3}\right)$

Then the base of the triangle lies on the $y-axis$ and has length $3+\frac{e^2}{3}$, and the height of the triangle is e since the lines meet at $x=e$

$\therefore$ Area $=\frac{1}{2}bh=\frac{1}{2}\left(3+\frac{e^2}{3}\right)e=\frac{e}{2}\left(3+\frac{e^2}{3}\right)$

10 $\int_1^3\frac{(2x-3)\sqrt{x^2-3x+3}}{x^2-3x+3}dx=\int_1^3\frac{2x-3}{\sqrt{x^2-3x+3}}dx$

Let $u=x^2-3x+3\Rightarrow du=(2x-3)dx$

Limits: $x=3\Rightarrow u=3,\ x=1\Rightarrow u=1$

$\therefore\int_1^3\frac{(2x-3)\sqrt{x^2-3x+3}}{x^2-3x+3}dx=\int_1^3u^{-\frac{1}{2}}du=2\left[u^{\frac{1}{2}}\right]_1^3$

$=2\left(\sqrt{3}-1\right)$

11 Let $2x = \sin\theta \Rightarrow dx = \frac{1}{2}\cos\theta d\theta$

Limits: $x = \frac{1}{2} \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$,

$x = a \Rightarrow \sin\theta = 2a \Rightarrow \theta = \arcsin(2a)$

$\therefore \int_a^{\frac{1}{2}} \frac{1}{\sqrt{1-4x^2}}dx = \int_{\frac{\pi}{2}}^{\arcsin(2a)} \frac{1}{\sqrt{1-\sin^2\theta}}\left(\frac{1}{2}\cos\theta\right)d\theta$

$= \int_{\frac{\pi}{2}}^{\arcsin(2a)} \frac{1}{2}d\theta = \left[\frac{\theta}{2}\right]_{\frac{\pi}{2}}^{\arcsin(2a)}$

$= \frac{1}{2}\left(\arcsin(2a) - \frac{\pi}{2}\right) = \frac{\pi}{24}$

$\Rightarrow \arcsin(2a) = \frac{7\pi}{12}$

$\Rightarrow 2a = \sin\frac{7\pi}{12}$

$\therefore a = \frac{1}{2}\sin\left(\frac{7\pi}{12}\right) = \frac{1}{2}\sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$

$= \frac{1}{2}\left(\sin\frac{\pi}{3}\cos\frac{\pi}{4} + \cos\frac{\pi}{3}\sin\frac{\pi}{4}\right)$

$= \frac{1}{2}\left(\frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}}\right)$

$= \frac{\sqrt{3}+1}{4\sqrt{2}}$

12 Any tangent passing through the origin is of the form $y = mx$

and furthermore, any such line will intersect $f(x)$ once. Therefore, the tangent at the point of contact satisfies the equations

$mx = e^{\frac{x}{2}-1}$ and $m = f'(x) = \frac{1}{2}e^{\frac{x}{2}-1}$

$\Rightarrow \frac{1}{2}xe^{\frac{x}{2}-1} = e^{\frac{x}{2}-1} \Rightarrow x = 2$

$\Rightarrow m = \frac{1}{2}e^{\frac{2}{2}-1} = \frac{1}{2}$

$\therefore y = \frac{1}{2}x$

13 Let $3x = 2\cos u \Rightarrow 3dx = -2\sin u\ du$

$\therefore \int \sqrt{4-9x^2}dx = \int \sqrt{4-9\left(\frac{2}{3}\cos u\right)^2}\left(-\frac{2}{3}\sin u\right)du$

$= -\frac{2}{3}\int \sin u\sqrt{4-4\cos^2 u}du$

$= -\frac{4}{3}\int \sin^2 u du$ (using the identity $\cos 2u = 1 - 2\sin^2 u$)

$= -\frac{2}{3}\int (1-\cos 2u)du$

$$= -\frac{2}{3}\left(u - \frac{1}{2}\sin 2u\right) + C$$

$$= -\frac{2}{3}(u - \sin u \cos u) + C$$

$$= \frac{2}{3}\sin u \cos u - \frac{2}{3}u + C$$

$$= \frac{2}{3}\sqrt{1 - \frac{9x^2}{4}}\left(\frac{3x}{2}\right) - \frac{2}{3}\arccos\left(\frac{3x}{2}\right) + C$$

$$= \frac{x}{2}\sqrt{4 - 9x^2} - \frac{2}{3}\arccos\left(\frac{3x}{2}\right) + C$$

14 By parts twice:

Let $u = x^2$ $\quad dv = e^{\frac{1}{2}}$

$du = 2x$ $\quad v = 2e^{\frac{1}{2}}$

$\int x^2 e^{\frac{x}{2}} dx = 2x^2 e^{\frac{x}{2}} - 4\int x e^{\frac{x}{2}} dx$

Integrating by parts again

Let $u = x$ $\quad dv = e^{\frac{x}{2}}$

$du = dx$ $\quad v = 2e^{\frac{x}{2}}$

$$= 2x^2 e^{\frac{x}{2}} - 4\left(2xe^{\frac{x}{2}} - 2\int e^{\frac{x}{2}} dx\right)$$

$$= 2x^2 e^{\frac{x}{2}} - 8xe^{\frac{x}{2}} + 16e^{\frac{x}{2}} + C$$

$$= 2e^{\frac{x}{2}}\left(x^2 - 4x + 8\right) + C$$

15 Let $I = \int 3^x \sin x dx$

Integrating by parts:

Let $u = 3^x$ $\quad dv = \sin x$

$du = (\ln 3)3^x$ $\quad v = -\cos x$

$I = \int 3^x \sin x dx = -3^x \cos x + \ln 3\int 3^x \cos x dx$

Integrating by parts again:

Let $u = 3^x$ $\quad dv = \cos x$

$du = (\ln 3)3^x$ $\quad v = \sin x$

$$I = -3^x \cos x + \ln 3\left(3^x \sin x - \ln 3\int 3^x \sin x dx\right)$$

$$\Rightarrow I = -3^x \cos x + (\ln 3)3^x \sin x - (\ln 3)^2 I$$

$$\Rightarrow \left(1 + (\ln 3)^2\right) I = 3^x\left((\ln 3)\sin x - \cos x\right)$$

$$\therefore I = \frac{3^x}{1 + (\ln 3)^2}\left((\ln 3)\sin x - \cos x\right)$$

Exam-style questions

16 a Attempt to factorise (1 mark)

$(3e^x - 1)(e^x + 4) = 0$ (1 mark)

$e^x = \frac{1}{3} \Rightarrow x = \ln\left(\frac{1}{3}\right)$ (1 mark)

$= -\ln 3$ (1 mark)

$e^x = -4$ has no solutions (1 mark)

b Attempt to factorise (1 mark)

$(\ln x - 9)(\ln x + 4) = 0$ (1 mark)

$\ln x = 9 \Rightarrow x = e^9$ (1 mark)

$\ln x = -4 \Rightarrow x = e^{-4}$ (1 mark)

17 a $10^{5x-1} = 15$ (1 mark)

$\log_{10} 10^{5x-1} = \log_{10} 15$ (1 mark)

$5x - 1 = \log_{10} 15$ (1 mark)

$x = \frac{1 + \log_{10} 15}{5}$ (1 mark)

b $\ln\left(3^{2-x}\right) = \ln\left(7^{\frac{x}{2}}\right)$ (1 mark)

$(2 - x)\ln 3 = \frac{x}{2}\ln 7$ (1 mark)

$2\ln 3 - x\ln 3 = \frac{x\ln 7}{2}$ (1 mark)

$4\ln 3 - 2x\ln 3 = x\ln 7$

$4\ln 3 = x\ln 7 + 2x\ln 3$

$4\ln 3 = x(\ln 7 + 2\ln 3)$

$x = \frac{4\ln 3}{(\ln 7 + 2\ln 3)}$ (1 mark)

18 a $y = 3\log_{10}(2x + 100)$ (1 mark)

$y = 3\log_{10}[2(x + 50)]$ (1 mark)

The transformations required are therefore:

Translation 50 units to the left (1 mark)

Stretch along the x-axis, scale factor $\frac{1}{2}$ (1 mark)

Stretch along the y-axis, scale factor 3 (1 mark)

b $y = \log_{10}(2x+100)^3$

$= \dfrac{\ln(2x+100)^3}{\ln 10}$ (use of change of base formula) (1 mark)

$= \left(\dfrac{3}{\ln 10}\right)\ln(2x+100)$ (1 mark)

$\dfrac{dy}{dx} = \left(\dfrac{3}{\ln 10}\right)\left(\dfrac{2}{2x+100}\right)$ (2 marks)

$= \dfrac{6}{(2x+100)\ln 10}$ (or equivalent) (1 mark)

19 a $\log_{16} 4 = \dfrac{\log_4 4}{\log_4 16} = \dfrac{1}{\log_4 16}$ (1 mark)

$= \dfrac{1}{\log_4 4^2}$ (1 mark)

$= \dfrac{1}{2\log_4 4}$

$= \dfrac{1}{2}$ (1 mark)

b $\log_{16}(x-4) - \log_{16}(x-12) = \dfrac{1}{2}$

$\log_{16}(x-4) - \log_{16}(x-12) = \log_{16} 4$ (1 mark)

$\log_{16}\left(\dfrac{x-4}{x-12}\right) = \log_{16} 4$ (1 mark)

$\dfrac{x-4}{x-12} = 4$ (1 mark)

$x - 4 = 4x - 48$

$3x = 44$

$x = \dfrac{44}{3}$ (1 mark)

20 a $\dfrac{dy}{dx} = -x^3e^{-x} + 3x^2e^{-x}$ (2 marks)

$\left(= e^{-x}\left(3x^2 - x^3\right)\right)$

Substituting $x = 3$ gives $\dfrac{dy}{dx} = 0$, hence a stationary point. (1 mark)

$$\frac{d^2y}{dx^2} = e^{-x}\left(6x - 3x^2\right) - e^{-x}\left(3x^2 - x^3\right)$$ (2 marks)

$$= e^{-x}\left(6x - 6x^2 + x^3\right)$$

Substituting $x = 3$ (1 mark)

$\frac{d^2y}{dx^2} = -9e^{-3} < 0$, so a maximum. (1 mark)

b $x = 1 \Rightarrow y = \frac{1}{e}$ (1 mark)

$x = 1 \Rightarrow \frac{dy}{dx} = \frac{2}{e}$ (1 mark)

Equation of tangent is $y - \frac{1}{e} = \frac{2}{e}(x-1)$ (2 marks)

$ey - 1 = 2x - 2$

$2x - ey - 1 = 0$ (1 mark)

21 $2x^2 + 3x - 2 = (2x-1)(x+2)$ (1 mark)

$$\frac{5x}{(2x-1)(x+2)} = \frac{A}{2x-1} + \frac{B}{x+2}$$

$5x = A(x+2) + B(2x-1)$ (1 mark)

$x = \frac{1}{2} \Rightarrow A = 1$ (1 mark)

$x = -2 \Rightarrow B = 2$ (1 mark)

$$\int_1^5 \left(\frac{1}{2x-1} + \frac{2}{x+2}\right) dx = \left[\frac{1}{2}\ln|2x-1| + 2\ln|x+2|\right]_1^5$$ (2 marks)

$= \frac{1}{2}\ln 9 + 2\ln 7 - 2\ln 3$ (2 marks)

$= \ln 3 + 2\ln 7 - 2\ln 3$ (1 mark)

$= 2\ln 7 - \ln 3$

$= \ln 49 - \ln 3$

$= \ln\left(\frac{49}{3}\right)$ (1 mark)

So $p = \frac{49}{3}$

22 a $\int_{\frac{1}{6}}^{\frac{1}{3}} \frac{dx}{\sqrt{1-9x^2}} = \frac{1}{3}\int_{\frac{1}{6}}^{\frac{1}{3}} \frac{dx}{\sqrt{\frac{1}{9} - x^2}}$ (1 mark)

Substitute $x = \frac{1}{3}\sin u$ (1 mark)

$\frac{dx}{du} = \frac{1}{3}\cos u$ (1 mark)

$\frac{1}{3}\int_{\frac{1}{6}}^{\frac{1}{3}} \frac{dx}{\sqrt{\frac{1}{9} - x^2}} = \frac{1}{9}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos u \ du}{\sqrt{\frac{1}{9} - \frac{1}{9}\sin^2 u}}$ (1 mark)

$= \frac{1}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos u \ du}{\sqrt{1 - \sin^2 u}}$

$= \frac{1}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} du$ (1 mark)

$= \frac{1}{3}\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$ (2 marks)

$= \frac{1}{3}\left(\frac{\pi}{3}\right)$

$= \frac{\pi}{9}$ (1 mark)

b 0.349 (1 mark)

23 a $\int \frac{x}{\sin^2 x} \ dx = \int x \operatorname{cosec}^2 x \ dx$ (1 mark)

Use integration by parts with $u = x$ and $\frac{dv}{dx} = \operatorname{cosec}^2 x$

$u = x \Rightarrow \frac{du}{dx} = 1$ (1 mark)

$\frac{dv}{dx} = \operatorname{cosec}^2 x \Rightarrow v = -\cot x$ (1 mark)

$\int \frac{x}{\sin^2 x} \ dx = -x\cot x + \int \cot x \ dx$ (2 marks)

$= -x\cot x + \ln|\sin x| + c$ (1 mark)

b $A = \left[-x\cot x + \ln|\sin x|\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$ (1 mark)

$= \left[-\frac{\pi}{2}\cot\frac{\pi}{2} + \ln\left|\sin\frac{\pi}{2}\right|\right] - \left[-\frac{\pi}{4}\cot\frac{\pi}{4} + \ln\left|\sin\frac{\pi}{4}\right|\right]$ (1 mark)

$= [-0 + 0] - \left[-\frac{\pi}{4} + \ln\frac{1}{\sqrt{2}}\right]$ (2 marks)

$= \frac{\pi}{4} - \ln\frac{1}{\sqrt{2}}$

$= \frac{\pi}{4} - \ln 2^{-\frac{1}{2}}$ (1 mark)

$= \frac{\pi}{4} + \frac{1}{2}\ln 2$ (1 mark)

$= \frac{\pi}{4} + \frac{1}{4}\ln 4$ (1 mark)

$= \frac{1}{4}(\pi + \ln 4)$

24 $a = \log_2 343$ (1 mark)

$a + 3d = \log_2 1331$ (1 mark)

Solve simultaneously to find d (1 mark)

$\log_2 343 + 3d = \log_2 1331$

$3d = \log_2 1331 - \log_2 343$

$3d = \log_2\left(\frac{1331}{343}\right)$

$d = \frac{1}{3}\log_2\left(\frac{1331}{343}\right)$ (1 mark)

$d = \log_2\left(\frac{1331}{343}\right)^{\frac{1}{3}}$ (1 mark)

$d = \log_2\left(\frac{11}{7}\right)$ (1 mark)

So $\log_2 x = a + d$ (1 mark)

$= \log_2 343 + \log_2\left(\frac{11}{7}\right)$

$= \log_2\left(343 \times \frac{11}{7}\right)$ (1 mark)

$= \log_2(49 \times 11)$

$= \log_2 539$ (1 mark)

So $x = 539$

25 Use integration by parts

$u = e^{-x} \Rightarrow \frac{du}{dx} = -e^{-x}$ (1 mark)

$\frac{dv}{dx} = \sin 3x \Rightarrow v = -\frac{1}{3}\cos 3x$ (1 mark)

$$\int e^{-x}\sin 3x\ \mathrm{d}x = -\frac{e^{-x}\cos 3x}{3} - \int -\frac{1}{3}\cos 3x\left(-e^{-x}\right)\mathrm{d}x$$ (2 marks)

$$= -\frac{e^{-x}\cos 3x}{3} - \frac{1}{3}\int e^{-x}\cos 3x\ \mathrm{d}x$$

By using integration by parts a second time,

$$\int e^{-x}\cos 3x\ \mathrm{d}x = \frac{e^{-x}\sin 3x}{3} + \frac{1}{3}\int e^{-x}\sin 3x\ \mathrm{d}x$$ (2 marks)

So $\int e^{-x}\sin 3x\ \mathrm{d}x = -\frac{e^{-x}\cos 3x}{3} - \frac{1}{3}\left[\frac{e^{-x}\sin 3x}{3} + \frac{1}{3}\int e^{-x}\sin 3x\ \mathrm{d}x\right]$ (1 mark)

$$\int e^{-x}\sin 3x\ \mathrm{d}x = -\frac{e^{-x}\cos 3x}{3} - \frac{e^{-x}\sin 3x}{9} - \frac{1}{9}\int e^{-x}\sin 3x\ \mathrm{d}x$$

$$\frac{10}{9}\int e^{-x}\sin 3x\ \mathrm{d}x = -\frac{e^{-x}\cos 3x}{3} - \frac{e^{-x}\sin 3x}{9}$$ (1 mark)

$$\int e^{-x}\sin 3x\ \mathrm{d}x = \frac{9}{10}\left(-\frac{e^{-x}\cos 3x}{3} - \frac{e^{-x}\sin 3x}{9}\right)$$ (1 mark)

$$= -\frac{e^{-x}}{10}\left(\sin 3x + 3\cos 3x\right) + c$$

8 Modelling change: more calculus

Skills check

1 a $\lim_{x\to -4}\frac{x^2-16}{x+4}=\lim_{x\to -4}\frac{(x+4)(x-4)}{x+4}=\lim_{x\to -4}x-4=-4-4=-8$

b $\lim_{x\to 0}\frac{3x^3+x^2}{x^2}=\lim_{x\to 0}3x+1=0+1=1$

c $\lim_{x\to 0}\frac{1}{x^2+1}=\frac{1}{0^2+1}=1$

2 a $\lim_{x\to\infty}\frac{2x^4-3}{2+3x^4}=\lim_{x\to\infty}\frac{2-\frac{3}{x^4}}{\frac{2}{x^4}+3}=\frac{2-0}{0+3}=\frac{2}{3}$, horizontal asymptote is $y=\frac{2}{3}$

b $\lim_{x\to\infty}\frac{3x}{x^3-1}=3\lim_{x\to\infty}\frac{x}{x^3-1}=3\lim_{x\to\infty}\frac{\frac{1}{x^2}}{1-\frac{1}{x^3}}=3\left(\frac{0}{1-0}\right)=3(0)=0$, horizontal asymptote is $y=0$

c $\lim_{x\to\infty}\frac{(x-5)^2}{x^2-5}=\lim_{x\to\infty}\frac{x^2-10x+25}{x^2-5}=\lim_{x\to\infty}\frac{1-\frac{10}{x}+\frac{25}{x^2}}{1-\frac{5}{x^2}}=\frac{1-0+0}{1-0}=\frac{1}{1}=1$, horizontal asymptote is $y=1$

3 a Substitute $u=x^2-1\to du=2x\,dx$

$$\int 2x\sqrt{x^2-1}dx\Rightarrow\int\sqrt{u}du=\frac{2u^{\frac{3}{2}}}{3}+C$$

$$\Rightarrow\frac{2(x^2-1)^{\frac{3}{2}}}{3}+C$$

b Substitute $u=\sin x\to dx=\frac{1}{\cos x}du$

$$\int\sin x\cos x dx\Rightarrow\int u du=\frac{u^2}{2}+C$$

$$\Rightarrow\frac{\sin^2 x}{2}+C$$

c Integrate by parts $\int uv'=uv-\int vu'$

$$u=\ln x\Rightarrow u'=\frac{1}{x}$$

$$v'=4x+5\Rightarrow v=2x^2+5x$$

$$\Rightarrow(2x^2+5x)\ln x-\int\frac{2x^2+5x}{x}dx=(2x^2+5x)\ln x-\int 2x+5dx$$

$$=(2x^2+5x)\ln x-x^2-5x+C$$

$$=x((2x+5)\ln x-x-5)+c$$

Exercise 8A

1 **a** Find the points of intersection:

$x = -1, 0, 1$

$$A = \int_{-1}^{0}\left[\left(\frac{x^3}{2} + 2x^2 + 2x - \frac{1}{2}\right) - \left(-\frac{1}{2} + 3x + 2x^2 - \frac{x^3}{2}\right)\right]dx +$$
$$\int_{0}^{1}\left[\left(-\frac{1}{2} + 3x + 2x^2 - \frac{x^3}{2}\right) - \left(\frac{x^3}{2} + 2x^2 + 2x - \frac{1}{2}\right)\right]dx$$

$$A = 0.25 + 0.25 = \frac{1}{2}$$

b Find the points of intersection

$x = -1, 0, 1$

$$A = 2\int_{0}^{1}\left(x^{\frac{2}{3}} - x\right)dx$$

$$A = 2(0.1) = \frac{1}{5}$$

c Find the points of intersection

$x = -2, 0, 2$

$$A = \int_{-2}^{0}\left[\left(2x^3 - x^2 - 5x\right) - \left(3x - x^2\right)\right]dx + \int_{0}^{2}\left[\left(3x - x^2\right) - \left(2x^3 - x^2 - 5x\right)\right]dx$$
$$A = \int_{-2}^{0}\left(2x^3 - 8x\right)dx + \int_{0}^{2}\left(8x - 2x^3\right)dx$$
$$A = 8 + 8 = 16$$

d Find the points of intersection

$x = -0.934$ and 0.934

$$A = \int_{-0.934}^{0.934}\left(8\cos^2 x - \sec^2 x\right)dx \approx 8.59$$

e Find points of intersection

$x = \frac{\pi}{2}$

$$A = \int_{0}^{\frac{\pi}{2}}\left(1 - \tan\left(\frac{x}{2}\right)\right)dx = \left[x + 2\ln\left(\left|\cos\left(\frac{x}{2}\right)\right|\right)\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - \ln 2$$

2 **a** Find points of intersection

$x = 0.531$ and 3.43

$$A = \int_{0.531}^{3.43}\left(\frac{1}{1+x} - \left((x-2)^4 - 4\right)\right)dx \approx 10.1$$

b Point of intersection

$x = 0.476$

$$A = \int_{0.476}^{4} \left(\sqrt{x} - \left(e^{1-x} - 1\right)\right) dx \approx 7.00$$

c Points of intersection

x = 0.537, 2.58, 7.31 and 7.97

$$A = \int_{0.537}^{2.58} \left(2\sin x - \left(e^{\frac{x}{2}-4} + 1\right)\right) dx + \int_{7.31}^{7.97} \left(2\sin x - \left(e^{\frac{x}{2}-4} + 1\right)\right) dx \approx 1.34$$

3 Points of intersection

x = 1.39, 2.60 and 3.40

$$A = \int_{1.39}^{2.60} \left(\ln(x-1) - \cos 2x\right) dx + \int_{2.60}^{3.40} \left(\cos 2x - \ln(x-1)\right) dx \approx 0.528 + 0.14 \approx 0.668$$

Exercise 8B

1 The graphs of the functions taken two at a time intersect at the points (0,0), (1,4), and (2,1). The area of the region defined by these points of intersection is:

$$A = \int_{0}^{1} \left[\left(x^2 + 3x\right) - \left(x - \frac{x^2}{4}\right)\right] dx + \int_{1}^{2} \left[\frac{4}{x^2} - \left(x - \frac{x^2}{4}\right)\right] dx$$

$$= \int_{0}^{1} \left(\frac{5}{4}x^2 + 2x\right) dx + \int_{1}^{2} \left(\frac{4}{x^2} - x + \frac{x^2}{4}\right) dx$$

$$= \left[\frac{5x^3}{12} + x^2\right]_0^1 + \left[-\frac{4}{x} - \frac{x^2}{2} + \frac{x^3}{12}\right]_1^2$$

$$= \frac{17}{12} + \left(\frac{7}{12} + \frac{1}{2}\right) = \frac{5}{2} \text{ sq units}$$

2 The graphs of the functions taken two at a time intersect at the points (0,1), (0.5,2) and (0.631,1.56).

The area of the region ABC is:

$$A = \int_{0}^{0.5} \left(4^x - 2^x\right) dx + \int_{0.5}^{0.641} \left(\frac{1}{x} - 2^x\right) dx$$

$$= \left[\frac{4^x}{\ln 4} - \frac{2^x}{\ln 2}\right]_0^{0.5} + \left[\ln x - \frac{2^x}{\ln 2}\right]_{0.5}^{0.641}$$

$$= 0.1237\ldots + 0.03894\ldots$$

$$\approx 0.163 \text{ sq units}$$

3 The graphs of the functions intersect at the points (0,2), (0.925,2.29), (2.20,4.76) and (3.23,57.1).

The sum of the areas enclosed by the two graphs is:

$$A = \int_{0}^{0.925} \left[\left(x^4 - 7x^2 + 6x + 2\right) - \left(1 + x + e^{x^2-2x}\right)\right] dx + \int_{0.925}^{2.20} \left[\left(1 + x + e^{x^2-2x}\right) - \left(x^4 - 7x^2 + 6x + 2\right)\right] dx$$

$$= 0.842 + 0.241$$

$$= 1.083 \text{ sq units (to 3 dp)}$$

$$A = \int_0^{0.925} \left[\left(x^4 - 7x^2 + 6x + 2\right) - \left(1 + x + e^{x^2-2x}\right)\right]dx + \int_{0.925}^{2.20} \left[\left(1 + x + e^{x^2-2x}\right) - \left(x^4 - 7x^2 + 6x + 2\right)\right]dx +$$

$$\int_{2.20}^{3.23} \left[\left(x^4 - 7x^2 + 6x + 2\right) - \left(1 + x + e^{x^2-2x}\right)\right]dx$$

$= 0.842 + 2.406 + 8.224$

$= 11.5$ sq units (to 3 sf)

4 a Find the point of intersection

$x^3 = 4x \Rightarrow x\left(x^2 - 4\right) = x\left(x-2\right)\left(x+2\right) = 0$

$x \geq 0 \Rightarrow x = 0, 2$

The entire area enclosed is

$$A = \int_0^2 \left(4x - x^3\right)dx = \left[2x^2 - \frac{x^4}{4}\right]_0^2 = 4$$

$$\therefore \left[2x^2 - \frac{x^4}{4}\right]_0^k = 2$$

$$\Rightarrow 2k^2 - \frac{k^4}{4} = 2$$

$\Rightarrow k^4 - 8k^2 + 8 = 0$

$\Rightarrow k = 1.08$

b $y = x^3 \Rightarrow x = y^{\frac{1}{3}}$

$$y = 4x \Rightarrow x = \frac{y}{4}$$

$$\therefore \int_0^m \left(y^{\frac{1}{3}} - \frac{y}{4}\right)dy = 2$$

$$\Rightarrow \frac{3}{4}m^{\frac{4}{3}} - \frac{1}{8}m^2 = 2$$

$\Rightarrow m^2 - 6m^{\frac{4}{3}} + 16 = 0$

$\Rightarrow m = 2\sqrt{2} = 2.83$

Exercise 8C

1 a $V = \pi\int_1^4 y^2 dx = \pi\int_1^4 x dx = \frac{\pi}{2}\left[x^2\right]_1^4 = \frac{15\pi}{2}$

b $V = \pi\int_1^2 x^2 dy = \pi\int_1^2 y^4 dy = \frac{\pi}{5}\left[y^5\right]_1^2 = \frac{31\pi}{5}$

2 a $V = \pi\int_0^1 y^2 dy = \pi\int_0^1 \left(x^2 + x\right)^2 dx = \pi\int_0^1 \left(x^4 + 2x^3 + x^2\right)dx$

$$\pi\left[\frac{1}{5}x^5 + \frac{1}{2}x^4 + \frac{1}{3}x^3\right]_0^1 = \frac{31\pi}{30}$$

b $V = \pi\int_1^4 \left(1 - \sqrt{x}\right)^2 dx = \pi\int_1^4 \left(1 - 2\sqrt{x} + x\right)dx$

$$\pi\left[x-\frac{4}{3}x^{\frac{3}{2}}+\frac{1}{2}x^2\right]_1^4=\pi\left(\frac{4}{3}-\frac{1}{6}\right)=\frac{7\pi}{6}$$

c $V=\pi\int_0^{\frac{\pi}{2}}y^2dx=\pi\int_0^{\frac{\pi}{2}}\cos x\sin xdx=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\sin 2xdx$

$$=-\frac{\pi}{4}\left[\cos 2x\right]_0^{\frac{\pi}{2}}=-\frac{\pi}{4}\left[-1-1\right]=\frac{\pi}{2}$$

d $V=\pi\int_0^{\frac{\pi}{4}}y^2dx=\pi\int_0^{\frac{\pi}{4}}\tan^2 xdx=\pi\int_0^{\frac{\pi}{4}}\left(\sec^2 x-1\right)dx=$

$$\pi\left[\tan x-x\right]_0^{\frac{\pi}{4}}=\pi\left(1-\frac{\pi}{4}\right)=\frac{\pi\left(4-\pi\right)}{4}$$

e $V=\pi\int_0^1 y^2dx=\pi\int_0^1\left(e^{-x}\right)^2dx=\pi\int_0^1 e^{-2x}dx$

$$-\frac{\pi}{2}\left[e^{-2x}\right]_0^1=-\frac{\pi}{2}\left[e^{-2}-1\right]=\frac{\pi}{2}\left(1-\frac{1}{e^2}\right)$$

f $V=\pi\int_0^3 y^2dx=\pi\int_0^3\frac{1}{x+1}dx$

$$\pi\left[\ln\left(x+1\right)\right]_0^3=\pi\ln 4$$

3 $V=\pi\int_0^6 y^2dx$

$$=\pi\int_0^6\frac{x^2}{144}\left(36-x^2\right)dx$$

$$=\frac{\pi}{144}\int_0^6\left(36x^2-x^4\right)dx$$

$$=\frac{\pi}{144}\left[12x^3-\frac{1}{5}x^5\right]_0^6$$

$$=\frac{\pi}{2^4 3^2}\left[\left(2^5 3^4\right)-\frac{1}{5}\left(2^5 3^5\right)\right]$$

$$=\pi\left(18-\frac{1}{5}(54)\right)=\frac{36\pi}{5}$$

4 **a** $V=\pi\int_0^2 x^2dy$

$$=\pi\int_0^2 4\sin^2 ydy=2\pi\int_0^2\left(1-\cos 2y\right)dy$$

$$=2\pi\left[y-\frac{1}{2}\sin 2y\right]_0^2$$

$$=2\pi\left(2-\frac{1}{2}\sin 4\right)=\pi\left(4-\sin 4\right)$$

b $V=\pi\int_0^1 x^2dy$

$$= \pi\int_0^1 \tan^2\left(\frac{\pi y}{4}\right)dy$$

$$= \pi\int_0^1 \left(\sec^2\frac{\pi y}{4} - 1\right)dy$$

$$= \pi\left[\frac{4}{\pi}\tan\frac{\pi y}{4} - y\right]_0^1$$

$$= \pi\left(\frac{4}{\pi} - 1\right) = 4 - \pi$$

5 This is equivalent to rotating x = sin y from y = 0 to $y = \frac{\pi}{2}$ through 2π radians about the y-axis

$$\therefore V = \pi\int_0^{\frac{\pi}{2}} \sin^2 y dy = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} (1 - \cos 2y)dy$$

$$= \frac{\pi}{2}\left[y - \frac{1}{2}\sin 2y\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi^2}{4}$$

6 This is equivalent to rotating x = cos y from y = 0 to $y = \frac{\pi}{2}$ through 2π radians about the y-axis

$$\therefore V = \pi\int_0^{\frac{\pi}{2}} \cos^2 y dy = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} (1 + \cos 2y)dy$$

$$= \frac{\pi}{2}\left[y + \frac{1}{2}\sin 2y\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi^2}{4}$$

7 Intersection points: (0,0) and (2.31,1.16)

a $V = \pi\int_0^{2.31} \left((\arctan(x))^2 - \left(e^{\frac{x}{3}} - 1\right)^2\right)dx \approx 2.35$

b $V = \pi\int_0^{1.16} \left((3\ln(y+1))^2 - (\tan(y))^2\right) \approx 4.18$

8 The functions intersect at x = 0.467 and x = 2.10. Therefore,

$$V = \pi\int_{0.467}^{2.10} \left((1 + \ln x)^2 - \left(\tan\frac{x}{2}\right)^2\right)dx \approx 3.58$$

9 The functions intersect at x = 0.601, Therefore, $V = \pi\int_0^{0.601} \left((\cos x)^2 - (e^x - 1)^2\right)dx \approx 1.31$

Exercise 8D

1 **a** $s(t) = \int v(t)dt$

$$\int_0^2 (\sin 2t - t\sin 2t)dt$$
$$= -\frac{1}{2}[\cos 2t]_0^2 - \int_0^2 t\sin 2t dt$$
$$= \left[-\frac{1}{2}\cos 2t + \frac{t}{2}\cos 2t\right]_0^2 - \int_0^2 \frac{1}{2}\cos 2t dt$$
$$= \left[\frac{t-1}{2}\cos 2t - \frac{1}{4}\sin 2t\right]_0^2$$
$$= \frac{1}{2}\cos 4 - \frac{1}{4}\sin 4 + \frac{1}{2} \approx 0.362$$

b $D = \int_0^2 |(1-t)\sin 2t| dt \approx 0.479$

2 a i $3\cos t = 0 \Rightarrow t = \frac{\pi}{2}, \frac{3\pi}{2}$

ii $3\cos t < 0 \Rightarrow \frac{\pi}{2} < t < \frac{3\pi}{2}$

iii $3\cos t > 0 \Rightarrow 0 < t < \frac{\pi}{2}$ and $\frac{3\pi}{2} < t < 2\pi$

b $\int_0^{2\pi} 3\cos t dt = 0$

c $\int_0^{2\pi} |3\cos t| dt = 12\int_0^{\frac{\pi}{2}} \cos t dt = 12[\sin t]_0^{\frac{\pi}{2}} = 12$

3 a i $\sin 2t = 0 \Rightarrow t = 0, \frac{\pi}{2}$

ii $\sin 2t < 0 \Rightarrow$ no solution

iii $0 < t < \frac{\pi}{2}$

b $\int_0^{\frac{\pi}{2}} \sin 2t dt = -\frac{1}{2}[\cos 2t]_0^{\frac{\pi}{2}} = -\frac{1}{2}(-1-1) = 1$

c $\int_0^{\frac{\pi}{2}} |\sin 2t| dt = \int_0^{\frac{\pi}{2}} \sin 2t dt = 1$

4 a i $\sqrt{2-t} = 0 \Rightarrow t = 2$

ii $\sqrt{2-t} < 0 \Rightarrow$ no solution

iii $\sqrt{2-t} > 0 \Rightarrow 0 < t < 2$

b $\int_0^2 \sqrt{2-t} dt = \left[-\frac{2}{3}(2-t)^{\frac{3}{2}}\right]_0^2 = \frac{4\sqrt{2}}{3}$

c $\int_0^2 \left|\sqrt{2-t}\right| dt = \int_0^2 \sqrt{2-t}\,dt = \frac{4\sqrt{2}}{3}$

5 a i $e^{\cos t}\sin t = 0 \Rightarrow t = 0, \pi, 2\pi$

ii $e^{\cos t}\sin t < 0 \Rightarrow \pi < t < 2\pi$

iii $e^{\cos t}\sin t > 0 \Rightarrow 0 < t < \pi$

b $\int_0^{2\pi} e^{\cos t}\sin t\,dt = \left[-e^{\cos t}\right]_0^2 = 0$

c $\int_0^{2\pi} \left|e^{\cos t}\sin t\right| dt = \int_0^{\pi} e^{\cos t}\sin t\,dt - \int_{\pi}^{2\pi} e^{\cos t}\sin t\,dt$

$= -\left[e^{\cos t}\right]_0^{\pi} + \left[e^{\cos t}\right]_{\pi}^{2\pi}$

$= -\left(\frac{1}{e} - e\right)\left(e - \frac{1}{e}\right)$

$= 2\left(e - \frac{1}{e}\right) \approx 4.70$

6 a i $\frac{t}{1+t^2} = 0 \Rightarrow t = 0$

ii $\frac{t}{1+t^2} < 0 \Rightarrow$ no solution

iii $\frac{t}{1+t^2} > 0 \Rightarrow 0 < t \le 3$

b $\int_0^3 \frac{t}{1+t^2} = \frac{1}{2}\left[\ln\left(1+t^2\right)\right]_0^3 = \frac{1}{2}\ln 10 \approx 1.51$

c $\int_0^3 \left|\frac{t}{1+t^2}\right| dt = \int_0^3 \frac{t}{1+t^2}dt = \frac{1}{2}\ln 10 \approx 1.51$

7 $\int_0^t 3e^{-\frac{t}{3}}dt = -9\left[e^{-\frac{t}{3}}\right]_0^t = 9\left(1 - e^{-\frac{t}{3}}\right)$

8 a $\int_0^{t_2} v(t)dt = s(t_1) - s(0) = s(t_1) - 8 = -2 \Rightarrow s(t_1) = 6$

b $\int_{t_1}^{t_2} v(t)dt = s(t_2) - s(t_1) = s(t_2) - 6 = 16 \Rightarrow s(t_2) = 22$

c $s(t_1) = 6$, so 6cm to the right of the origin at $t = t_1$

$s(t_2) = 22$, so 22cm to the right of the origin at $t = t_2$

$\int_{t_1}^{t_2} v(t)dt = s(t_3) - s(t_2) = s(t_3) - 22 = -7 \Rightarrow s(t_3) = 15$

so 15cm to the right of the origin at $t = t_3$

9 a Area of first triangle: $\frac{1}{2}\left(\frac{3}{2}\right)(3)=\frac{9}{4}$

Area of trapezium: $\frac{3+2}{2}(1)=\frac{5}{2}$

Area of second triangle: $\frac{1}{2}\left(\frac{5}{2}\right)(3)=\frac{15}{4}$

$\therefore s(7)=s(0)+\int_0^7 v(t)dt$

$=-2-\frac{9}{4}+\frac{5}{2}-\frac{15}{4}=-\frac{22}{4}=-5.5$

So 5.5cm to the left of the origin

b $\int_0^7 |v(t)|dt=\frac{9}{4}+\frac{5}{2}+\frac{15}{4}=\frac{17}{2}=8.5$

Exercise 8E

1 a $\frac{dy}{dx}=\frac{2x^2}{y^2}\Rightarrow y^2\frac{dy}{dx}=2x^2$

$\therefore \int y^2dy=\int 2x^2dx$

$\Rightarrow \frac{y^3}{3}=\frac{2x^3}{3}+c$

$\Rightarrow y=\left(2x^3+c\right)^{\frac{1}{3}}=\sqrt[3]{2x^3+c}$

b $\frac{dy}{dx}=\mathrm{e}^{y-x}\Rightarrow \mathrm{e}^{-y}\frac{dy}{dx}=\mathrm{e}^{-x}$

$\therefore \int \mathrm{e}^{-y}dy=\int \mathrm{e}^{-x}dx$

$\Rightarrow -\mathrm{e}^{-y}=-\mathrm{e}^{-x}+c$

$\Rightarrow \mathrm{e}^{-y}=\mathrm{e}^{-x}+c$

$\Rightarrow -y=\ln\left(\mathrm{e}^{-x}+c\right)$

$\Rightarrow y=-\ln\left(\mathrm{e}^{-x}+c\right)$

c $\mathrm{e}^{-2x}\frac{fy}{dx}=\frac{3}{y}\Rightarrow y\frac{dy}{dx}=3\mathrm{e}^{2x}$

$\therefore \int ydy=\int 3\mathrm{e}^{2x}dx$

$\Rightarrow \frac{y^2}{2}=\frac{3}{2}\mathrm{e}^{2x}+c$

$\Rightarrow y^2=3\mathrm{e}^{2x}+c$

$\Rightarrow y=\pm\sqrt{3\mathrm{e}^{2x}+c}$

d $\frac{dy}{dx}=y\cos x\Rightarrow \frac{1}{y}\frac{dy}{dx}=\cos x$

$$\int \frac{1}{y} dy = \int \cos x dx$$
$$\Rightarrow \ln y = \sin x + c$$
$$\Rightarrow y = e^{\sin x + c} = A e^{\sin x}$$

e $\frac{dy}{dx} = \frac{x + \sin x}{5y^4} \Rightarrow 5y^4 \frac{dy}{dx} = x + \sin x$

$$\therefore \int 5y^4 dy = \int (x + \sin x) dx$$
$$\Rightarrow y^5 = \frac{1}{2}x^2 - \cos x + c$$
$$\Rightarrow y = \left(\frac{1}{2}x^2 - \cos x + c\right)^{\frac{1}{5}} = \sqrt[5]{\frac{x^2}{2} - \cos x + c}$$

f $(1 + 9x^2)\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{1 + 9x^2}$

$$\therefore \int dy = \int \frac{1}{1 + 9x^2} dx$$
$$\Rightarrow y = \frac{1}{3}\arctan(3x) + c$$

2 a $x^{\frac{1}{2}} + y^{\frac{1}{2}} \frac{dy}{dx} = 0 \Rightarrow y^{\frac{1}{2}} \frac{dy}{dx} = -x^{\frac{1}{2}}$ subject to $y(1) = 4$

$$\therefore \int y^{\frac{1}{2}} dy = -\int x^{\frac{1}{2}} dx$$
$$\Rightarrow \frac{2}{3}y^{\frac{3}{2}} = -\frac{2}{3}x^{\frac{3}{2}} + c$$
$$\Rightarrow y^{\frac{3}{2}} = -x^{\frac{3}{2}} + c$$

When $x = 1$, $y = 4$

$$\Rightarrow 8 = -1 + c \Rightarrow c = 9$$
$$\therefore y^{\frac{3}{2}} = 9 - x^{\frac{3}{2}}$$
$$\Rightarrow y = \left(9 - x^{\frac{3}{2}}\right)^{\frac{2}{3}} = \sqrt[3]{\left(9 - x^{\frac{3}{2}}\right)^2}$$

b $x e^{x^2} + y \frac{dy}{dx} = 0 \Rightarrow y \frac{dy}{dx} = -x e^{x^2}$ subject to $y(0) = 1$

$$\therefore \int y dy = -\int x e^{x^2} dx$$
$$\Rightarrow \frac{1}{2}y^2 = -\frac{1}{2}e^{x^2} + c$$
$$\Rightarrow y^2 = -e^{x^2} + c$$

At $x = 0, y = 1$

$\therefore 1 = -1 + c \Rightarrow c = 2$

$\therefore y^2 = 2 - e^{-x^2}$

$\Rightarrow y = \sqrt{2 - e^{-x^2}}$ (take positive root due to initial conditions)

c $\frac{dy}{dx} = e^{x - 2y} \Rightarrow e^{2y} \frac{dy}{dx} = e^x$ subject to $y(0) = 0$

$\therefore \int e^{2y} dy = \int e^{x} dx$

$\Rightarrow \frac{1}{2} e^{2y} = e^{x} + c$

When $x = 0, y = 0$

$\therefore \frac{1}{2} = 1 + c \Rightarrow c = -\frac{1}{2}$

$\Rightarrow \frac{1}{2} e^{2y} = e^{x} - \frac{1}{2}$

$\Rightarrow e^{2y} = 2e^{x} - 1$

$\Rightarrow y = \frac{1}{2} \ln\left(2e^{x} - 1\right)$

d $\frac{dy}{dx} = 2xy \sin\left(x^2\right) \Rightarrow \frac{1}{y}\frac{dy}{dx} = 2x \sin\left(x^2\right)$ subject to $y(0) = 1$

$\therefore \int \frac{1}{y} dy = \int 2x \sin\left(x^2\right) dx$

$\Rightarrow \ln y = -\cos\left(x^2\right) + c$

When $x = 0, y = 1$

$\therefore 0 = -1 + c \Rightarrow c = 1$

$\Rightarrow \ln y = 1 - \cos\left(x^2\right)$

$y = e^{1-\cos\left(x^2\right)}$

e $e^{x} \frac{dy}{dx} + x = 0 \Rightarrow \frac{dy}{dx} = -x e^{-x}$ subject to $y(0) = 2$

$\therefore \int dy = -\int x e^{-x} dx$

$\Rightarrow y = -\left[\left(-x e^{-x}\right) + \int e^{-x} dx\right] = x e^{-x} + e^{-x} + c = e^{-x}\left(x + 1\right) + c$

$\therefore$ When $x = 0, y = 2$

$\Rightarrow 2 = 1 + c \Rightarrow c = 1$

$\therefore y = x e^{-1} + e^{-x} + 1 = e^{-x}\left(x + 1\right) + 1$

Exercise 8F

1 a $\frac{dT}{dt} = -k\left(T - T_0\right) = k\left(100 - T\right)$

b $\frac{1}{T - T_0}\frac{dT}{dt} = -k$

$\ln\left(T - T_0\right) = -kt + c$

$\Rightarrow T = T_0 + A e^{-kt}$

When $t = 0, T = 100$

$100 = T_0 + A \qquad (1)$

When $t = 10, T = 80$

$80 = T_0 + A e^{-10k} \qquad (2)$

When $t = 20, T = 65$

$65 = T_0 + A e^{-20k} \qquad (3)$

Multiplying (2) by e^{10k} and subtracting from (1)

$100 - 80e^{10k} = T_0\left(1 - e^{10k}\right) \Rightarrow T_0 = \dfrac{100 - 80e^{10k}}{1 - e^{10k}}$

$\therefore T_0 = \dfrac{80 - 65e^{10k}}{1 - e^{10k}} = \dfrac{100 - 80e^{10k}}{1 - e^{10k}}$

$\Rightarrow 100 - 80e^{10k} = 80 - 65e^{10k} \Rightarrow k = \dfrac{1}{10}\ln\left(\dfrac{4}{3}\right)$

$\Rightarrow T_0 = 20°\text{C}$

c $T_0 = 20 \Rightarrow A = 80$

$\therefore T(t) = 20 + 80e^{-\frac{1}{10}\ln\left(\frac{4}{3}\right)t}$

$\therefore T(30) = 20 + 80e^{-3\ln\left(\frac{4}{3}\right)} \approx 53.8°$

2 **a** $\dfrac{dT}{dt} = k(180 - T)$

b $\dfrac{1}{180 - T}\dfrac{dT}{dt} = k$

$\therefore -\ln(180 - T) = kt + c$

$\Rightarrow T(t) = 180 - Ae^{-kt}$

$T(0) = 180 - A = 0 \Rightarrow A = 180$

$T(5) = 180\left(1 - e^{-5k}\right) = 120$

$1 - e^{-5k} = \dfrac{2}{3} \Rightarrow k = -\dfrac{1}{5}\ln\left(\dfrac{1}{3}\right) = \dfrac{1}{5}\ln 3$

$\therefore T(t) = 180\left(1 - e^{-\frac{t}{5}\ln 3}\right)$

so $T(10) = 180\left(1 - e^{-2\ln 3}\right) = 160°\text{C}$

3 Since $OP = PQ$, $OQ = 2x$ and the coordinates of Q are $(2x, 0)$

The gradient of PQ is therefore $\dfrac{y - 0}{x - 2x} = -\dfrac{y}{x}$

Since PQ is the tangent of P, its gradient is $\dfrac{dy}{dx}$

$\therefore \dfrac{dy}{dx} = -\dfrac{y}{x}$

$\Rightarrow \dfrac{dy}{y} = -\dfrac{dx}{x}$

$\Rightarrow \int \dfrac{dy}{y} = -\int \dfrac{dx}{x}$

$\Rightarrow \ln y = -\ln x + c$, since $x > 0$ and $y > 0$

$\Rightarrow \ln y + \ln x = c$

$\Rightarrow \ln(xy) = c$

$\Rightarrow xy = k$

The point (1,2) lies on the curve

$\Rightarrow k = 2$

$\therefore y = \dfrac{2}{x}$

4 a $\dfrac{dP}{dt} = kP$

b $\dfrac{1}{P}\dfrac{dP}{dt} = k$

$\Rightarrow \int \dfrac{1}{P}dP = \int kdt$

$\Rightarrow P = P_0 e^{kt}$ where P_0 is the initial population size

$P(0) = 500 \Rightarrow P_0 = 500$

$\therefore P = 500 e^{kt}$

When $t = 3, p = 10000$

$\therefore 10000 = 500 e^{3k} \Rightarrow k = \dfrac{\ln 20}{3}$

$\therefore P = 500 e^{\frac{\ln 20}{3}t}$

c $P(5) = 500 e^{\frac{5\ln 20}{3}} = 73680.63\ldots \approx 73700$

d When $P = 500000$

$500000 = 500 e^{\frac{\ln 20}{3}t}$

$\Rightarrow t = \dfrac{3}{\ln 20}\ln(1000) = 6.917596\ldots$

so 7 hours (to the nearest hour)

5 a $\dfrac{dV}{dr} = cr$

$V = \dfrac{4}{3}\pi r^3$

$\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$

$\Rightarrow cr = 4\pi r^2 \dfrac{dr}{dt}$

$\Rightarrow \dfrac{dr}{dt} = \dfrac{c}{4\pi r} = \dfrac{k}{r} \Rightarrow k = \dfrac{c}{4\pi}$

b $\dfrac{dr}{dt} = \dfrac{k}{r} \Rightarrow \int rdr = \int kdt$ □

$\dfrac{r^2}{2} = kt + d$

Substituting $(0, 8)$, $d = 32$

$\dfrac{r^2}{2} = kt + 32$

Substituting $(30, 12)$, $k = \dfrac{4}{3}$

$t = 15 \Rightarrow \frac{r^2}{2} = \frac{4}{3} \bullet 15 + 32$

$\Rightarrow r^2 = 104$

$r = 10\text{cm}$

6 **a** $\frac{dN}{dt} = kN(100 - N)$

$\Rightarrow \frac{1}{N(100-N)}\frac{dN}{dt} = k$

$\frac{1}{N(100-N)} = \frac{a}{N} + \frac{b}{100-N}$

$\Rightarrow 1 = a(100 - N) + bN$

$N = 0 : 1 = 100a \Rightarrow a = \frac{1}{100}$

$N = 100 : 1 = 100b \Rightarrow b = \frac{1}{100}$

$\therefore \frac{1}{100}\left[\frac{1}{N} + \frac{1}{100-N}\right]\frac{dN}{dt} = k$

$\Rightarrow \frac{1}{100}\left[\ln N - \ln(100 - N)\right] = kt + c$

$\Rightarrow \ln\left(\frac{N}{100-N}\right) = 100kt + c$

$\Rightarrow \frac{N}{100-N} = A\,e^{100kt}$

b $A\,e^{100k} = \frac{1}{99}$

$A\,e^{200k} = \frac{1}{49}$

$\Rightarrow e^{100k} = \frac{A\,e^{200k}}{A\,e^{100k}} = \frac{99}{49}$

$k = \frac{1}{100}\ln\frac{99}{49} = 0.0070.. = 0.007\,(3\text{ dp})$

$\therefore Ae^{\ln\frac{99}{49}} = \frac{1}{99} \Rightarrow A = \frac{49}{9801} = 0.004999.. = 0.005\ (3\text{ dp})$

c Rearranging the result from the first part

$$N(t) = \frac{100A\,e^{100kt}}{1 + A\,e^{100kt}} = \frac{\frac{4900}{9801}e^{\ln\left(\frac{99}{49}\right)t}}{1 + \frac{49}{9801}e^{\ln\left(\frac{99}{49}\right)t}}\left(\approx \frac{100(2.02)^t}{2.02^t + 200}\right)$$

$N(7)$ = 40.725… so 41 students

7 $\frac{dP}{dt} = \frac{P}{50}\left(1 - \frac{P}{2000}\right) = \frac{P}{100000}(2000 - P)$

$$\Rightarrow \frac{100000}{P(2000-P)}\frac{dP}{dt} = 1$$

$$\Rightarrow \left(\frac{50}{P} + \frac{50}{2000-P}\right)\frac{dP}{dt} = 1$$

$$\Rightarrow 50\ln P - 50\ln(2000-P) = t + c$$

$$\Rightarrow \frac{P}{2000-P} = A\,e^{\frac{t}{50}}$$

$$\Rightarrow P(t) = \frac{2000A\,e^{\frac{t}{50}}}{1 + A\,e^{\frac{t}{50}}}$$

$$P(0) = 300$$

$$\Rightarrow \frac{2000A}{1+A} = 300 \Rightarrow A = \frac{3}{17}$$

$$\therefore P(t) = \frac{6000\,e^{\frac{t}{50}}}{17 + 3\,e^{\frac{t}{50}}}\left(\approx \frac{2000(1.02)^t}{1.02^t + 5.67}\right)$$

$P(3) = 315.6234\ldots$

so 316 to the nearest integer

8 a $\dfrac{dP}{dt} = \dfrac{4}{5}P\left(1 - \dfrac{P}{350}\right) = \dfrac{2}{875}P(350-P)$

$$\therefore \frac{1}{P(350-P)}\frac{dP}{dt} = \frac{2}{875}$$

$$\Rightarrow \left(\frac{1}{P} + \frac{1}{350-P}\right)\frac{dP}{dt} = \frac{2(350)}{875} = \frac{700}{875} = \frac{4}{5}$$

$$\Rightarrow \ln\left(\frac{P}{350-P}\right) = \frac{4}{5}t + c$$

$$\Rightarrow \frac{P}{350-P} = A\,e^{\frac{4}{5}t}$$

$$\Rightarrow P(t) = \frac{350A\,e^{\frac{4}{5}t}}{1 + A\,e^{\frac{4}{5}t}}$$

$$P(0) = 7 \Rightarrow A = \frac{1}{49}$$

$$\therefore P(t) = \frac{350\,e^{\frac{4}{5}t}}{49 + e^{\frac{4}{5}t}}\left(\approx \frac{350(2.226)^t}{2.226^t + 49}\right)$$

b $\lim\limits_{t\to\infty} P(t) = 350$

9 a $a = \dfrac{dv}{dt} = -\dfrac{1}{2}v$

b $\dfrac{dv}{dt} = -\dfrac{1}{2}v$

$\therefore \frac{1}{v}\frac{dv}{dt} = -\frac{1}{2}$

$\Rightarrow \int \frac{1}{v}dv = \int -\frac{1}{2}dt$

$\Rightarrow \ln v = -\frac{t}{2} + c$

$\Rightarrow v(t) = Ae^{-\frac{t}{2}}$

If $v(0) = 20$, then $A = 20$

so $v(t) = 20e^{-\frac{t}{2}}$

10 a $a = v\frac{dv}{ds} = \frac{2s}{s^2+1}$

$\Rightarrow \int v dv = \int \frac{2s}{s^2+1}ds$

$\Rightarrow \frac{v^2}{2} = \ln(s^2+1) + c$

When $s = 1$, $v = 2$

$\therefore 2 = \ln 2 + c \Rightarrow c = 2 - \ln 2$

$\therefore v^2 = 2\ln(s^2+1) + 4 - 2\ln 2$

$v = \sqrt{2\ln(s^2+1) + 4 - 2\ln 2}$ (positive sqaure root due to initial conditions)

$= \sqrt{2\ln\left(\frac{s^2+1}{2}\right) + 4}$

b When $s = 5$,

$v = \sqrt{2\ln(26) + 4 - 2\ln 2} = \sqrt{\ln\left(\frac{676}{4}\right) + 4} = \sqrt{\ln 169 + 4} = \sqrt{2\ln 13 + 4} \approx 3 \text{ m s}^{-1}$

11 $a = \frac{dv}{dt} = -\left(\frac{v^2}{50} + 32\right) = -\frac{1}{50}(v^2 + 1600)$

$\therefore \frac{1}{1600+v^2}\frac{dv}{dt} = -\frac{1}{50}$

$\Rightarrow \int \frac{1}{1600+v^2}dv = -\int \frac{1}{50}dt$

$\Rightarrow \frac{1}{40}\arctan\left(\frac{v}{40}\right) = -\frac{t}{50} + c$

$\Rightarrow \arctan\left(\frac{v}{40}\right) = -\frac{4t}{5} + c$

$\Rightarrow v(t) = 40\tan\left(c - \frac{4t}{5}\right)$

$v(0) = 40$

$\Rightarrow 40 = 40\tan c \Rightarrow \tan c = 1 \Rightarrow c = \frac{\pi}{4}$

$\therefore v(t) = 40\tan\left(\frac{\pi}{4} - \frac{4t}{5}\right)$

$v(10) = -53.8 \text{ m s}^{-1}$ (3 s.f.)

Exercise 8G

1 **a** $xy' = x^2 \cos x + y$

$\Rightarrow x(v + xv') = x^2 \cos x + xv$
$\Rightarrow x^2 v' = x^2 \cos x$
$\Rightarrow v' = \cos x$
$v = \sin x + c$
$\Rightarrow y = x(\sin x + c)$

b $x^2 y' = 3x^2 - xy$

$\Rightarrow x^2(v + xv') = 3x^2 - x^2 v$
$\Rightarrow v + xv' = 3 - v$
$\Rightarrow xv' = 3 - 2v$
$\Rightarrow \dfrac{1}{3-2v} v' = \dfrac{1}{x}$
$\Rightarrow \int \dfrac{1}{3-2v} dv = \int \dfrac{1}{x} dx$
$\Rightarrow -\dfrac{1}{2}\ln(3-2v) = \ln x + c$
$\Rightarrow \ln(3-2v) = c - 2\ln x$
$\Rightarrow 3 - 2v = A e^{-2\ln x} = \dfrac{A}{x^2}$

$\Rightarrow v = \dfrac{1}{2}\left(3 - \dfrac{A}{x^2}\right)$
$\Rightarrow y = \dfrac{x}{2}\left(3 - \dfrac{A}{x^2}\right) = \dfrac{1}{2x}(3x^2 - A) = \dfrac{3x^2 + c}{2x}$

c $x^2 \dfrac{dy}{dx} = y^2 + xy + 4x^2$

$x^2(v + xv') = x^2 v^2 + x^2 v + 4x^2$
$\Rightarrow xv' = v^2 + 4$
$\Rightarrow \dfrac{1}{4+v^2} v' = \dfrac{1}{x}$
$\Rightarrow \int \dfrac{1}{4+v^2} dv = \int \dfrac{1}{x} dx$
$\Rightarrow \dfrac{1}{2}\arctan\left(\dfrac{v}{2}\right) = \ln x + c$
$\Rightarrow v = 2\tan(2\ln x + c)$
$\Rightarrow y = 2x\tan(2\ln x + c)$

2 Let $v = \dfrac{y}{x}$

$x^2 y' = y^2 + xy + 4x^2$

From **1 c** we know the solution is

$y = 2x\tan(2\ln x + c)$

When $x = 1, y = 2$ so

$2 = 2\tan c \Rightarrow \tan c = 1 \Rightarrow c = \frac{\pi}{4}$

$\therefore y = 2x\tan\left(2\ln x + \frac{\pi}{4}\right)$

3 $y' = \frac{y}{x} + \frac{y^2}{x^2}$ subject to $y(1) = 2$

Let $v = \frac{y}{x}$

$\therefore xv' + v = v + v^2$

$\Rightarrow xv' = v^2$

$\Rightarrow \frac{1}{v^2}v' = \frac{1}{x}$

$\Rightarrow \int \frac{1}{v^2}dv = \int \frac{1}{x}dx$

$\Rightarrow -\frac{1}{v} = \ln x + c$

$\Rightarrow v = -\frac{1}{c + \ln x}$

$y = -\frac{x}{c + \ln x}$

When $x = 1, y = 2$

$\therefore 2 = -\frac{1}{c} \Rightarrow c = -\frac{1}{2}$ so

$y = \frac{2x}{1 - 2\ln x}$

4 $x^2y' = y^2 + 3xy + 2x^2$ subject to $y(1) = -1$

$\Rightarrow x^2(v + xv') = x^2v^2 + 3x^2v + 2x^2$

$\Rightarrow v + xv' = v^2 + 3v + 2$

$\Rightarrow xv' = v^2 + 2v + 2$

$\Rightarrow \frac{1}{v^2 + 2v + 2}v' = \frac{1}{x}$

$\Rightarrow \int \frac{1}{1 + (v+1)^2}dv = \int \frac{1}{x}dx$

$\Rightarrow \arctan(v + 1) = \ln x + c$

$\Rightarrow v = \tan(\ln x + c) - 1$

$y = x(\tan(\ln x + c) - 1)$

When $x = 1, y = -1$

$\therefore -1 = \tan c - 1 \Rightarrow c = 0$

$\Rightarrow y = x(\tan(\ln x) - 1)$

Exercise 8H

1 **a** $y' + y = e^x$

$I = e^{\int dx} = e^x$

$\Rightarrow e^x y' + e^x y = e^{2x}$

$\Rightarrow \frac{dy}{dx}\left(e^x y\right) = e^{2x}$

$\Rightarrow e^x y = \frac{1}{2}e^{2x} + c$

$\Rightarrow y = \frac{1}{2}e^x + c\,e^{-x}$

b $(x-1)y' = y - x^2y \Rightarrow (x-1)y' + \left(x^2-1\right)y = 0$

$\Rightarrow y' + \frac{x^2-1}{x-1}y = y' + (x+1)y = 0$

$I = e^{\int (x+1)dx} = e^{\left(\frac{x^2}{2}+x\right)}$

$\Rightarrow e^{\left(\frac{x^2}{2}+x\right)}\left(y' + (x+1)y\right) = 0$

$\Rightarrow \frac{d}{dx}\left(e^{\frac{x^2}{2}+x}\,y\right) = 0$

$e^{\frac{x^2}{2}+x}\,y = A$

$y = A\,e^{-\frac{x^2}{2}-x}$

c $xy' + y = x^2 + 1$

$\therefore \frac{d}{dx}(xy) = x^2 + 1$

$\Rightarrow xy = \frac{1}{3}x^3 + x + c$

$\Rightarrow y = \frac{1}{3}x^2 + 1 + \frac{c}{x} = \frac{x^3 + 3x + c}{3x}$

d $y' + y = \sin\left(e^x\right)$

$I = e^{\int dx} = e^x$

$e^x\left(y' + y\right) = e^x \sin\left(e^x\right)$

$\frac{d}{dx}\left(e^x y\right) = e^x \sin\left(e^x\right)$

$\Rightarrow e^x y = -\cos\left(e^x\right) + A$

$\Rightarrow y = -e^{-x}\,cox\left(e^x\right) + A\,e^{-x}$

e $y' + xy = x\,e^{x^2}$

$$I = e^{\int x dx} = e^{\frac{1}{2}x^2}$$

$$\therefore e^{\frac{x^2}{2}} y' + x e^{\frac{x^2}{2}} y = x e^{\frac{3x^2}{2}}$$

$$\Rightarrow \frac{d}{dx}\left(e^{\frac{x^2}{2}} y\right) = x e^{\frac{3x^2}{2}}$$

$$\Rightarrow e^{\frac{x^2}{2}} y = \frac{1}{3} e^{\frac{3x^2}{2}} + c$$

$$\Rightarrow y = \frac{1}{3} e^{x^2} + c e^{-\frac{x^2}{2}}$$

f $x^2 y' + 2xy = \cos x$

$$\frac{d}{dx}\left(x^2 y\right) = \cos x$$

$$\Rightarrow x^2 y = \sin x + c$$

$$\Rightarrow y = \frac{\sin x + c}{x^2}$$

g $xy' + 2y = \cos x \Rightarrow y' + \frac{2}{x} y = \frac{\cos x}{x}$

$$I = e^{\int \frac{2}{x} dx} = e^{2\ln x} = x^2$$

$$\therefore x^2 y' + 2xy = x\cos x$$

$$\Rightarrow \frac{d}{dx}\left(x^2 y\right) = x\cos x$$

$$\Rightarrow x^2 y = \int x \cos x dx$$

$$\Rightarrow x^2 y = x\sin x + \cos x + c$$

$$\Rightarrow y = \frac{\sin x}{x} + \frac{\cos x}{x^2} + \frac{c}{x^2}$$

2 **a** $xy' = y + x^3 \sin x \Rightarrow y' - \frac{1}{x} y = x^2 \sin x$ subject to $y(\pi) = 0$

$$I = e^{-\int \frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}$$

$$\therefore \frac{1}{x} y' - \frac{1}{x^2} y = x\sin x$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{x} y\right) = x\sin x$$

$$\Rightarrow \frac{y}{x} = \int x \sin x \, dx$$

$$\Rightarrow \frac{y}{x} = \sin x - x\cos x + c$$

$$\Rightarrow y = x\sin x - x^2\cos x + cx$$

When $x = \pi, y = 0$

$$\therefore 0 = \pi \sin \pi - \pi^2 \cos \pi + c\pi \Rightarrow c = -\pi$$

$$\Rightarrow y = x\sin x - x^2 \cos x - \pi x$$

b $xy' - \frac{3}{x} = 2y \Rightarrow y' - \frac{2}{x} y = \frac{3}{x^2}$ subject to $y(2) = 5$

$$I = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = \frac{1}{x^2}$$

$$\therefore \frac{1}{x^2}y' - \frac{2}{x^3}y = \frac{3}{x^4}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{x^2}y\right) = \frac{3}{x^4}$$

$$\Rightarrow \frac{y}{x^2} = \int \frac{3}{x^4}dx = -\frac{1}{x^3} + c$$

$$\Rightarrow y = cx^2 - \frac{1}{x}$$

When $x = 2, y = 5$

$$\Rightarrow 5 = 4c - \frac{1}{2} \Rightarrow c = \frac{11}{8}$$

$$\therefore y = \frac{11x^2}{8} - \frac{1}{x}$$

c $y' + y\tan x = \sec x$ subject to $y(0) = 2$

$$I = e^{\int \tan x dx} = e^{-\ln(\cos x)} = \sec x$$

$$\therefore \sec x\frac{dy}{dx} + \sec x \tan x y = \sec^2 x$$

$$\Rightarrow \frac{d}{dx}(y\sec x) = \sec^2 x$$

$$\Rightarrow y\sec x = \tan x + c$$

$$\Rightarrow y = \sin x + c\cos x$$

When $x = 0, y = 2$

$$\therefore 2 = c \Rightarrow c = 2$$

$$\Rightarrow y = \sin x + 2\cos x$$

Exercise 8I

1

n	x_n	y_n	$\frac{dy}{dx} = f(x_n, y_n) = 2 - 2y_n - e^{-4x_n}$
0	0	1	-1
1	0.1	0.9	-0.470320046
2	0.2	0.8529679954	-0.1552649549
3	0.3	0.8374414999	0.02392278827
4	0.4	0.8398337787	0.1184359246
5	0.5	0.8516773712	

so $y(0.5) \approx 0.852$

2

n	x_n	y_n	$\frac{dy}{dx} = f(x_n, y_n) = 2x_n y_n$
0	1	2	4
1	1.4	3.6	10.08
2	1.8	7.632	27.4752

3	2.2	18.62208	81.937152
4	2.6	51.3969408	267.2640922
5	3.0	158.3025777	

$y(3) = 158.3$

3

n	x_n	y_n	$\frac{dy}{dx} = f(x_n, y_n) = x_n^2 + y_n^2$
0	0	1	1
1	0.1	1.1	1.22
2	0.2	1.222	1.533284
3	0.3	1.3753284	1.981528208
4	0.4	1.573481221	

$\therefore y(0.4) \approx 1.57$, underestimated since y' is increasing

4

n	x_n	y_n	$\frac{dy}{dx} = f(x_n, y_n) = e^{x_n} + 2y_n^2$
0	0	1	3
1	0.1	1.3	4.485170918
2	0.2	1.748517092	7.336026799
3	0.3	2.482119772	

$y(0.3) \approx 2.48$

5 a

n	x_n	y_n	$\frac{dy}{dx} = f(x_n, y_n) = 2x_n(1 + x_n^2 - y_n)$
0	1	2	0
1	1.1	2	0.462
2	1.2	2.0462	0.94512
3	1.3	2.140712	

$\therefore y(1.3) \approx 2.14$, using a smaller step size would give a more accurate answer

b Integrating factor method

$$\frac{dy}{dx} + 2xy = 2x + 2x^3$$
$$I = e^{\int 2x dx} = e^{x^2}$$
$$\therefore e^{x^2}\frac{dy}{dx} + 2x\,e^{x^2}y = e^{x^2}\left(2x + 2x^3\right)$$
$$\Rightarrow \frac{dy}{dx}\left(e^{x^2}y\right) = e^{x^2}\left(2x + 2x^3\right)$$
$$\Rightarrow e^{x^2}y = \int 2x^3 e^{x^2} dx + \int 2x\,e^{x^2} dx$$

Note that $\int 2x^3 e^{x^2} dx = \int x^2\left(2x\,e^{x^2}\right)dx = x^2 e^{x^2} - \int 2x\,e^{x^2} dx$ (by parts)

$$\therefore e^{x^2}y = x^2 e^{x^2} - \int 2x\,e^{x^2} dx + \int 2x\,e^{x^2} dx + c$$
$$= x^2 e^{x^2} + c$$
$$\Rightarrow y = x^2 + c\,e^{-x^2}$$

When $x = 1, y = 2$

$$\therefore 2 = 1 + c\,e^{-1} \Rightarrow c = e$$
$$\Rightarrow y(x) = x^2 + e^{1-x^2}$$
$$y(1.3) = 2.191576$$

Exercise 8J

1 **a** $\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1} = 1$

b $\lim_{x\to 0}\frac{\tan 3x}{\tan 4x} = \lim_{x\to 0}\frac{3\sec^2 3x}{4\sec^2 4x} = \frac{3}{4}$

c $\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{\sin x}{1} = 0$

d $\lim_{x\to 3}\frac{e^{x^2} - e^9}{x-3} = \lim_{x\to 3}\frac{2x\,e^{x^2}}{1} = 6\,e^9$

e $\lim_{x\to e}\frac{1-\ln x}{\frac{x}{e} - 1} = \lim_{x\to e}\frac{-\frac{1}{x}}{\frac{1}{e}} = -1$

f $\lim_{x\to 1}\frac{\arctan x - \frac{\pi}{4}}{\tan\frac{\pi x}{4} - 1} = \lim_{x\to 1}\frac{\frac{1}{1+x^2}}{\frac{\pi}{4}\sec^2\frac{\pi x}{4}} = \frac{\frac{1}{2}}{\frac{\pi}{4}\left(\sqrt{2}\right)^2} = \frac{1}{\pi}$

g $\lim_{x\to 0}\frac{\cos x - \cos 2x}{x - \cos x} = \lim_{x\to 0}\frac{-\sin x + 2\sin 2x}{1 + \sin x} = 0$

h $\lim_{x\to 0}\frac{\sin\left(x^2\right)}{\ln\left(\cos x\right)} = \lim_{x\to 0}\frac{2x\cos\left(x^2\right)}{\tan x} = -2\lim_{x\to 0}\frac{\cos\left(x^2\right) - 2x^2\sin\left(x^2\right)}{\sec^2 x} = -2$

i $\lim_{x\to 0}\frac{\ln\left(1+x\right) - x}{\cos x - 1} = \lim_{x\to 0}\frac{\frac{1}{1+x} - 1}{-\sin x} = \lim_{x\to 0}\frac{x}{(1+x)\sin x} = \lim_{x\to 0}\frac{1}{(1+x)\cos x + \sin x} = 1$

j $\lim_{x\to\infty}\frac{x^2}{e^{1-x}} = \lim_{x\to\infty}\frac{2x}{-e^{1-x}} = \lim_{x\to\infty}\frac{2}{e^{1-x}} = 0$

k $\lim\limits_{x\to\infty}\frac{x^2}{e^x\ln x}=\lim\limits_{x\to\infty}\frac{2x}{e^x\left(\frac{1}{x}+\ln x\right)}=\lim\limits_{x\to\infty}\frac{2}{e^x\left(\frac{1}{x}+\ln x-\frac{1}{x^2}+\frac{1}{x}\right)}=\lim\limits_{x\to\infty}\frac{2}{e^x\left(\ln x+\frac{2}{x}-\frac{1}{x^2}\right)}=0$

l $\lim\limits_{x\to0}\frac{\cos x-\cos 2x}{x^2}=\lim\limits_{x\to0}\frac{-\sin x+2\sin 2x}{2x}=\lim\limits_{x\to0}\frac{-\cos x+4\cos 2x}{2}=\frac{-1+4}{2}=\frac{3}{2}$

2 a $\lim\limits_{x\to\infty}\frac{2x^3-x^2}{2-x-x^4}=\lim\limits_{x\to\infty}\frac{6x^2-2x}{-1-4x^3}=\lim\limits_{x\to\infty}\frac{12x-2}{-12x^2}=\lim\limits_{x\to\infty}\frac{12}{-24x}=0$

so $y = 0$

b $\lim\limits_{x\to\infty}\frac{3x^3-2x^2+1}{2-x-x^3}=\lim\limits_{x\to\infty}\frac{9x^2-4x}{-1-3x^2}=\lim\limits_{x\to\infty}\frac{18x-4}{-6x}=\lim\limits_{x\to\infty}\frac{18}{-6}=-3$

so $y = -3$

c $\lim\limits_{x\to\infty}\frac{3x^5-7x}{2x^2+4}=\lim\limits_{x\to\infty}\frac{15x^4-7}{4x}=\lim\limits_{x\to\infty}\frac{60x^3}{4}=\infty$

so no horizontal asymptote exists

d $\lim\limits_{x\to\infty}\frac{2^x}{x^2}=\lim\limits_{x\to\infty}\frac{(\ln 2)2^x}{2x}=\lim\limits_{x\to\infty}\frac{(\ln 2)^2 2^x}{2}=\infty$

so no horizontal asymptote exists

Exercise 8K

1 a $f(0)=0$

$f'(x)=e^x(x+1)\Rightarrow f'(0)=1$
$f''(x)=e^x(x+2)\Rightarrow f''(0)=2$
$f'''(x)=e^x(x+3)\Rightarrow f'''(0)=3$
$f^{(4)}(x)=e^x(x+4)\Rightarrow f^{(4)}(0)=4$
$\therefore f(x)=x+x^2+\frac{1}{2}x^3+\frac{1}{6}x^4+\ldots$

b $f(0)=1$

$f'(0)=-1$
$f''(0)=1$
$f'''(0)=-1$

$f^{(4)}(0)=1$
$\therefore f(x)=1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}+\ldots$

c $f(0)=0$

$f'(x) = \cos x \Rightarrow f'(0) = 1$
$f''(x) = -\sin x \Rightarrow f''(0) = 0$
$f'''(x) = -\cos x \Rightarrow f'''(0) = -1$
$f^{(4)}(x) = \sin x \Rightarrow f^{(4)}(0) = 0$
$\therefore f(x) = x - \frac{x^3}{6} + \ldots$

d $f(0) = 1$

$f'(x) = -2\sin x\cos x = -\sin 2x \Rightarrow f'(0) = 0$
$f''(x) = -2\cos 2x \Rightarrow f''(0) = -2$
$f'''(x) = 4\sin 2x \Rightarrow f'''(0) = 0$
$f^{(4)}(x) = 8\cos 2x \Rightarrow f^{(4)}(0) = 8$
$\therefore f(x) = 1 - x^2 + \frac{x^4}{3} + \ldots$

e $f(0) = 1$

$f'(x) = (1-x)^{-2} \Rightarrow f'(0) = 1$
$f''(x) = 2(1-x)^{-3} \Rightarrow f''(0) = 2$
$f'''(x) = 6(1-x)^{-4} \Rightarrow f'''(0) = 6$
$f^{(4)}(x) = 24(1-x)^{-5} \Rightarrow f^{(4)}(0) = 24$
$\therefore f(x) = 1 + x + x^2 + x^3 + x^4 + \ldots$

f $f(0) = 1$

$f'(x) = -2x(1+x^2)^{-2} \Rightarrow f'(0) = 0$
$f''(x) = -2(1+x^2)^{-2} + 8x^2(1+x^2)^{-3} \Rightarrow f''(0) = -2$
$f'''(x) = 8x(1+x^2)^{-3} + 16x(1+x^2)^{-3} +$ terms divisible by $x^2 \Rightarrow f'''(0) = 0$
$f^{(4)}(x) = 8(1+x^2)^{-3} + 16(1+x^2)^{-2} +$ terms divisble by $x \Rightarrow f^{(4)}(0) = 24$
$\therefore f(x) = 1 - x^2 + x^4 + \ldots$

g Using question **1 f**

$$f(x) = x\left(\frac{1}{1+x^2}\right) = x\left(1 - x^2 + x^4 + \ldots\right) = x - x^3 + \ldots$$

Exercise 8L

1 **a** $e^{3x} = 1 + 3x + \frac{9x^2}{2!} + \ldots + \frac{3^n x^n}{n!} + \ldots = \sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$

b $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots + (-1)^n x^n = \sum_{n=0}^{\infty} (-1)^n x^n$

c Using the series sound in part **b**
$\frac{1}{1+2x} = 1 - 2x + (2x)^2 - (3x)^3 + \ldots + (-1)^n (2x)^n + \ldots = \sum_{n=0}^{\infty} (-1)^n (2x)^n$

d $\arctan\left(x^2\right) = x^2 - \dfrac{x^6}{3} + \dfrac{x^{10}}{5} + \ldots + (-1)^n \dfrac{x^{2(2n+1)}}{2n+1} + \ldots = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2(2n+1)}}{2n+1}$

e $y = f(x) = \sin^2 x$

$f'(x) = 2\sin x \cos x = \sin 2x \Rightarrow f'(0) = 0$
$f''(x) = 2\cos 2x \Rightarrow f''(0) = 2$
$f'''(x) = -4\sin 2x \Rightarrow f'''(0) = 0$
$f^{(4)}(x) = -8x\cos 2x \Rightarrow f^{(4)}(0) = -8$

Can show inductively that $f^{(2m+1)}(0) = 0$ for $x \in \mathbb{N}$ and $f^{(2m)}(0) = (-1)^{m-1} 2^{2m-1}$

for $x \in \mathbb{N}$ so $f(x) = \sum_{m=1}^{\infty} \dfrac{(-1)^{m-1} 2^{2m-1}}{(2m)!} x^{2m}$

2 $\sin^2 x = \dfrac{1}{2}(1 - \cos 2x) = \dfrac{1}{2}\left(1 - \sum_{n=0}^{\infty} (-1)^n \dfrac{(2x)^{2n}}{(2n)!}\right)$

$= \dfrac{1}{2}\left(1 - 1 - \sum_{n=1}^{\infty} (-1)^n \dfrac{(2x)^{2n}}{(2n)!}\right)$

$= -\dfrac{1}{2}\sum_{n=1}^{\infty} (-1)^n \dfrac{2^{2n}}{(2n)!} x^{2n}$

$= \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{2^{2n-1}}{(2n)!} x^{2n}$

which coincides with the answer to **1 e**

3 $f(x) = \dfrac{7x - 2}{(x+1)(x-2)} = \dfrac{A}{x+1} + \dfrac{B}{x-2}$

$\Rightarrow 7x - 2 = A(x-2) + B(x+1)$
$x = 2 : 12 = 3B \Rightarrow B = 4$
$x = -1 : -9 = -3A \Rightarrow A = 3$

$\therefore \dfrac{7x-2}{(x+1)(x-2)} = \dfrac{3}{x+1} + \dfrac{4}{x-2}$

$= 3(x+1)^{-1} - 2\left(1 - \dfrac{x}{2}\right)^{-1}$

$= 3\sum_{n=0}^{\infty} (-1)^n x^n - 2\sum_{n=0}^{\infty} (-1)^n \left(-\dfrac{x}{2}\right)^n$

$= 3\sum_{n=0}^{\infty} (-1)^n x^n - 2\sum_{n=0}^{\infty} \left(-\dfrac{x}{2}\right)^n$

$= \sum_{n=0}^{\infty} \left(3(-1)^n - 2^{1-n}\right) x^n$

4 a $\lim_{x\to 0} \dfrac{\sin x - x}{x^3} = \lim_{x\to 0} \dfrac{\left(x - \dfrac{x^3}{6} + \dfrac{x^5}{120} - \ldots\right) - x}{x^3}$

$\lim_{x\to 0} \dfrac{\left(-\dfrac{x^3}{6} + \dfrac{x^5}{120} - \ldots\right)}{x^3} = \lim_{x\to 0}\left(-\dfrac{1}{6} + \dfrac{x^2}{120} - \ldots\right) = -\dfrac{1}{6}$

b $\lim\limits_{x\to 0}\dfrac{e^{x}-e^{-x}}{x}=\lim\limits_{x\to 0}\dfrac{\left(1+x+\dfrac{x^2}{2}+\dfrac{x^3}{6}+\ldots\right)-\left(1-x+\dfrac{x^2}{2}-\dfrac{x^3}{6}+\ldots\right)}{x}$

$$=\lim_{x\to 0}\frac{2\left(x+\dfrac{x^3}{6}+\dfrac{x^5}{120}+\ldots\right)}{x}=2\lim_{x\to 0}\left(1+\frac{x^2}{6}+\frac{x^4}{120}+\ldots\right)=2$$

c $\lim\limits_{x\to 0}\dfrac{(2-2\cos x)^3}{x^6}=8\lim\limits_{x\to 0}\dfrac{(1-\cos x)^3}{x^6}$

$$8\lim_{x\to 0}\frac{\left(1-\left(1-\dfrac{x^2}{2}+\dfrac{x^4}{24}+\ldots\right)\right)^3}{x^6}=8\lim_{x\to 0}\frac{\left(\dfrac{x^2}{2}-\dfrac{x^4}{24}+\ldots\right)^3}{x^6}$$

$$=8\lim_{x\to 0}\left(\frac{1}{2}-\frac{x^2}{24}+\ldots\right)^3=8\bullet\frac{1}{8}=1$$

d $\lim\limits_{x\to 0}\dfrac{x^2-\sin^2 x}{x^2\sin^2 x}=\lim\limits_{x\to 0}\dfrac{(x+\sin x)(x-\sin x)}{x^2\sin^2 x}$

$$=\lim_{x\to 0}\frac{\left(2x-\dfrac{x^3}{6}+\dfrac{x^5}{120}-\ldots\right)\left(\dfrac{x^3}{6}-\dfrac{x^5}{120}+\ldots\right)}{x^3-\dfrac{x^5}{6}+\dfrac{x^7}{120}-\ldots}$$

$$=\lim_{x\to 0}\frac{\left(2x-\dfrac{x^3}{6}+\dfrac{x^5}{120}-\ldots\right)\left(\dfrac{1}{6}-\dfrac{x^2}{120}+\ldots\right)}{1-\dfrac{x^2}{6}+\dfrac{x^4}{120}-\ldots}$$

$$\lim_{x\to 0}\frac{\dfrac{1}{3}+x(\ldots)}{1+x(\ldots)}=\frac{1}{3}$$

5 a $f(x)=\sqrt{1-x}=(1-x)^{\frac{1}{2}}=\sum\limits_{n=0}^{\infty}\dbinom{\frac{1}{2}}{n}(-x)^n$

$$\binom{\frac{1}{2}}{n}=\frac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\ldots\cdot\left(\frac{1}{2}-n+1\right)}{n!}$$

$$=(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-3)}{2^n n!}$$

$$=(-1)^{n-1}\frac{(2n)!}{(2^n n!)^2(2n-1)}$$

$$\Rightarrow\quad 1+\sum_{n=0}^{\infty}(-1)^{n-1}\frac{(2n)!}{(2^n n!)^2(2n-1)}(-1)^n x^n$$

$$\Rightarrow\sqrt{1-x}=1-\sum_{n=0}^{\infty}\frac{(2n)!}{(2^n n!)^2(2n-1)}x^n=1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3-\frac{5}{128}x^4-\ldots$$

b $f(x) = \dfrac{1}{(1+x)^3} = (1+x)^{-3} = \sum_{n=0}^{\infty}\binom{-3}{n}x^n$

$$\binom{-3}{n} = \frac{-3\cdot -4\cdot -5\cdot \ldots \cdot (3-n+1)}{n!}$$
$$= (-1)^n\frac{3\cdot 4\cdot 5\cdot \ldots \cdot (n+2)!}{2n!}$$
$$= (-1)^n\frac{(n+1)(n+2)}{2}$$
$$\therefore \frac{1}{(1+x)^3} = \sum_{n=0}^{\infty}(-1)^n\binom{n+2}{2}x^n = 1-3x+6x^2-10x^3+\ldots$$

c $f(x) = \dfrac{1}{\left(1-4x^2\right)^2} = \left(1-4x^2\right)^{-2} = \sum_{n=0}^{\infty}\binom{-2}{n}\left(-4x^2\right)^n$

$$\binom{-2}{n} = \frac{-2\cdot -3\cdot -4\cdot \ldots \cdot (2-n+1)}{n!}$$
$$= (-1)^n\frac{2\cdot 3\cdot 4\cdot \ldots \cdot (n+1)}{n!}$$
$$= (-1)^n\frac{(n+1)!}{n!} = (-1)^n(n+1)$$
$$\therefore \frac{1}{\left(1-4x^2\right)^2} = \sum_{n=0}^{\infty}(n+1)(-4)^n x^{2n} = \sum_{n=0}^{\infty}4^n(n+1)x^{2n}$$
$$= 1+8x^2+48x^4+\ldots$$

d $f(x) = \dfrac{1}{\sqrt[4]{1+2x^3}} = \left(1+2x^3\right)^{\frac{1}{4}} = \sum_{n=0}^{\infty}\binom{-\frac{1}{4}}{n}\left(2x^3\right)^n$

$$\binom{-\frac{1}{4}}{n} = \frac{-\frac{1}{4}\cdot -\frac{5}{4}\cdot -\frac{9}{4}\cdot \ldots \cdot \left(\frac{1}{4}-n+1\right)}{n!}$$
$$= (-1)^n\frac{1\cdot 5\cdot 9\cdot \ldots \cdot (4n-3)}{4^n n!}$$
$$\therefore \frac{1}{\sqrt[4]{1+2x^3}} = 1+\sum_{n=0}^{\infty}(-1)^n\frac{1\cdot 5\cdot 9\cdot \ldots \cdot (4n-3)}{4^n n!}\left(2x^3\right)^n$$
$$= 1+\sum_{n=0}^{\infty}(-1)^n\frac{1\cdot 5\cdot 9\cdot \ldots \cdot (4n-3)}{2^n n!}x^{3n}$$
$$= 1-\frac{1}{2}x^3+\frac{5}{8}x^6+\ldots$$

6 $f(x) = (1+x)^{\frac{1}{3}} = 1+\dfrac{1}{3}x+\dfrac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}x^2+\dfrac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}x^3$

$$+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{4!}x^4+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)\left(-\frac{11}{3}\right)}{5!}x^5+\ldots$$
$$= 1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3+\ldots$$

$f(0.2) \approx 1.06272$

7 $y(0) = -\frac{\pi}{2}$

$y'(x) = y\tan x + \cos x \Rightarrow y'(0) = 1$

$y''(x) = y'\tan x + y\sec x - \sin x \Rightarrow y''(0) = -\frac{\pi}{2}$

$\therefore y(x) = -\frac{\pi}{2} + x - \frac{\pi}{4}x^2 + \ldots$

8 $y(0) = 1$

$\therefore y'(x) = y^2 - x \Rightarrow y'(0) = 1^2 - 0 = 1$

$y''(x) = 2yy' - 1 \Rightarrow y''(0) = 1$

$y'''(x) = 2(y')^2 + 2yy'' \Rightarrow y'''(0) = 4$

$y^{(4)}(x) = 4y''y' + 2y'y'' + 2yy''' = 6y''y' + 2yy'''$

$\Rightarrow y^{(4)}(0) = 6 + 2(4) = 14$

$y^{(5)}(x) = 6y'''y' + 6(y'')^2 + 2y'y''' + 2yy^{(4)} = 8y'''y' + 6(y'')^2 + 2yy^{(4)}$

$\Rightarrow y^{(5)}(0) = 8(4) + 6 + 2(14) = 66$

$\therefore y(x) = 1 + x + \frac{x^2}{2!} + \frac{4x^3}{3!} + \frac{14x^4}{4!} + \frac{66x^5}{5!} + \ldots$

$y(0.2) = 1.2264$ (4 dp)

9 $y(0) = 0$

$\therefore y'(x) = y^2 + 1 \Rightarrow y'(0) = 0^2 + 1 = 1$

$y''(x) = 2yy' \Rightarrow y''(0) = 0$

$y'''(x) = 2(y')^2 + 2yy'' \Rightarrow y'''(0) = 2$

$y^{(4)}(x) = 4y''y' + 2y'y'' + 2yy''' = 6y''y' + 2yy'''$

$\Rightarrow y^{(4)}(0) = 0$

$y^{(5)}(x) = 6y'''y' + 6(y'')^2 + 2y'y''' + 2yy^{(4)} = 8y'''y' + 6(y'')^2 + 2yy^{(4)}$

$\Rightarrow y^{(5)}(0) = 8(2) = 16$

$\therefore y(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$

Which is the Maclurian expansion of $\tan x$.

Chapter review

1 **a** $A = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)dx$

$= \left[-\cos x - \sin x\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$

$= 2\left(\frac{2}{\sqrt{2}}\right) = 2\sqrt{2}$

b $A = \int_0^{\pi} (2\sin x - \sin 2x)dx$

$$= \left[-2\cos x + \frac{1}{2}\cos 2x\right]_0^{\pi} = 4$$

2 **a** $x^2 = x^3 \Rightarrow x^2(1-x) = 0 \Rightarrow x = 0, 1$

$$\therefore A = \int_0^1 (x^2 - x^3)dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{12}$$

b $|x| = x^{\frac{2}{3}}$

$$x > 0 \Rightarrow x = x^{\frac{2}{3}} \Rightarrow x^{\frac{2}{3}}\left(x^{\frac{1}{3}} - 1\right) = 0$$

$$\Rightarrow x = 0, 1$$

$$A = 2\int_0^1 \left(x^{\frac{2}{3}} - x\right)dx = 2\left[\frac{3}{5}x^{\frac{5}{3}} - \frac{x^2}{2}\right]_0^1 = \frac{1}{5}$$

c $x^4 - 2x^2 = 2x^2 \Rightarrow x^4 - 4x^2 = x^2(x^2 - 4) = x^2(x+2)(x-2) = 0$

$$\Rightarrow x = -2, 0, 2$$

From the sketch the graphs just touch at $x = 0$ so the enclosed area is the integral from $x = -2$ to $x = 2$

$$A = \int_{-2}^{2} \left[2x^2 - (x^4 - 2x^2)\right]dx = \int_{-2}^{2} (4x^2 - x^4)dx$$

$$= \left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_{-2}^{2} = \frac{128}{15}$$

d The graphs intersect at $x = 0$ and $x = 4$

$$\therefore A = 2\int_0^4 \left[\frac{x^2}{2} + 4 - |x^2 - 4|\right]dx$$

$$= 2\int_0^2 \left[\frac{x^2}{2} + 4 + x^2 - 4\right]dx + 2\int_2^4 \left[\frac{x^2}{2} + 4 - (x^2 - 4)\right]dx$$

$$= 2\int_0^2 \frac{3x^2}{2}dx + 2\int_2^4 \left(8 - \frac{x^2}{2}\right)dx$$

$$= \left[x^3\right]_0^2 + 2\left[8x - \frac{x^3}{2}\right]_2^4 = \frac{64}{3}$$

e Sketch the graphs and find the points of intersection for $x > 0$:

$$\sqrt{x} = \frac{x+6}{5} \Rightarrow 5\sqrt{x} = x + 6$$

$$\Rightarrow 25x = (x+6)^2 = x^2 + 12x + 36$$

$$\Rightarrow x^2 - 13x + 36 = (x-9)(x-4) = 0 \Rightarrow x = 4, 9$$

Substituting these values back into the equation verifies they are valid for $x < 0$:

$$\sqrt{-x} = \frac{x+6}{5} \Rightarrow -25x = (x+6)^2 = x^2 + 12x + 36$$

$$x^2 + 37x + 36 = (x+1)(x+36) = 0 \Rightarrow x = -1, -36$$

Substituting these into the original equations shows $x = -36$ is a spurious root and the only valid root in this region is x −1

$$\therefore A = \int_{-1}^{4}\left[\frac{x+6}{5} - \sqrt{|x|}\right]dx + \int_{4}^{9}\left[\sqrt{|x|} - \frac{x+6}{5}\right]dx$$

$$= \int_{-1}^{0}\left[\frac{x+6}{5} - \sqrt{-x}\right]dx + \int_{0}^{4}\left[\frac{x+6}{5} - \sqrt{x}\right]dx + \int_{4}^{9}\left[\sqrt{x} - \frac{x+6}{5}\right]dx$$

$$= \left[\frac{x^2}{10} + \frac{6x}{5} + \frac{2}{3}(-x)^{\frac{3}{2}}\right]_{-1}^{0} + \left[\frac{x^2}{10} + \frac{6x}{5} - \frac{2}{3}x^{\frac{3}{2}}\right]_{0}^{4} + \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^2}{10} - \frac{6x}{5}\right]_{4}^{9} = \frac{5}{3}$$

f The graphs intersect as $x = \pm 2$

Note that the total area is double the area enclosed in the first quadrant

$$\therefore A = 2\int_{0}^{2}\left(\frac{8}{4+x^2} - \frac{x^2}{4}\right)dx = 2\int_{0}^{2}\left(\frac{2}{1+\left(\frac{x}{2}\right)^2} - \frac{x^2}{4}\right)dx$$

$$= 2\left[4\arctan\left(\frac{x}{2}\right) - \frac{x^3}{12}\right]_{0}^{2}$$

$$= 2\left[4\arctan 1 - \frac{2}{3}\right] = 2\left[4\left(\frac{\pi}{4}\right) - \frac{2}{3}\right] = 2\pi - \frac{4}{3}$$

3 Total area enclosed is double the area enclosed in the first quadrant. The intersection in the first quadrant is at x = 1 (by inspection or algebraically)

$$\therefore A = 2\int_{0}^{1}\left(\frac{2x}{x^2+1} - x^3\right)dx$$

$$= 2\left[\ln\left(x^2+1\right) - \frac{x^4}{4}\right]_{0}^{1}$$

$$= 2\ln 2 - \frac{1}{2}$$

4 **a** $\dfrac{xy}{x+1} = \dfrac{dy}{dx}$

$$\frac{1}{y}\frac{dy}{dx} = \frac{x}{x+1} = \frac{x+1-1}{x+1} = 1 - \frac{1}{x+1}$$

$$\therefore \int \frac{1}{y}dy = \int\left(1 - \frac{1}{x+1}\right)dx$$

$$\therefore \ln y = x - \ln(x+1) + c$$

$$\Rightarrow y = e^{x-\ln(x+1)+c} = \frac{Ae^x}{x+1}$$

b $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{x}{x^2+1}$

$$\therefore \int \frac{1}{y}dy = \int \frac{x}{x^2+1}dx$$
$$\Rightarrow \ln y = \frac{1}{2}\ln\left(x^2+1\right)+c$$
$$\Rightarrow y = e^{\frac{1}{2}\ln\left(x^2+1\right)+c} = A\,e^{\ln\left(\sqrt{x^2+1}\right)} = A\sqrt{x^2+1}$$
$$\text{or } y = \sqrt{c\left(x^2+1\right)}$$

c $1+xy' = y^2 \Rightarrow x\dfrac{dy}{dx} = y^2-1 = (y+1)(y-1)$

$$\therefore \frac{1}{(y+1)(y-1)}\frac{dy}{dx} = \frac{1}{x}$$
$$\Rightarrow \frac{1}{2}\left(\frac{1}{y-1}-\frac{1}{y+1}\right)\frac{dy}{dx} = \frac{1}{x}$$
$$\Rightarrow \int \frac{1}{2}\left(\frac{1}{y-1}-\frac{1}{y+1}\right)dy = \int \frac{1}{x}dx$$
$$\Rightarrow \frac{1}{2}\ln\left(\frac{y-1}{y+1}\right) = \ln x + c$$
$$\Rightarrow \ln\left(\frac{y-1}{y+1}\right) = 2\ln x + c = \ln\left(x^2\right)+c$$
$$\Rightarrow \frac{y-1}{y+1} = Ax^2$$
$$\Rightarrow y = \frac{1+Ax^2}{1-Ax^2}$$
$$\text{or } y = \frac{cx^2+1}{1-cx^2}$$

d $\dfrac{dy}{dx} = \dfrac{1}{xy+y} = \dfrac{1}{y(x+1)}$

$$\therefore y\frac{dy}{dx} = \frac{1}{x+1}$$
$$\Rightarrow \int y\,dy = \int \frac{1}{x+1}dx$$
$$\Rightarrow \frac{y^2}{2} = \ln(x+1)+c$$
$$\Rightarrow y = \pm\sqrt{c+2\ln(x+1)}$$

e $\dfrac{dy}{dx} + \dfrac{2}{x}y + x^2 = \sin 2x$

$$I = e^{\int \frac{2}{x}dx} = x^2$$

$$\therefore x^2 \frac{dy}{dx} + 2xy = x^2 \sin 2x - x^4$$

$$\Rightarrow \frac{d}{dx}\left(x^2 y\right) = x^2 \sin 2x - x^4$$

$$\Rightarrow x^2 y = \int \left(x^2 \sin 2x - x^4\right)dx = \int x^2 \sin 2x dx - \frac{x^5}{5} + c$$

$$= -\frac{1}{2}x^2 \cos 2x + \int x \cos 2x dx - \frac{x^5}{5} + c$$

$$= -\frac{1}{2}x^2 \cos 2x + \frac{x}{2}\sin 2x - \frac{1}{2}\int \sin 2x dx - \frac{x^5}{5} + c$$

$$= \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x - \frac{x^2}{2}\cos 2x - \frac{x^5}{5} + c$$

$$\therefore y = \frac{\sin 2x}{2x} + \frac{\cos 2x}{4x^2} - \frac{\cos 2x}{2} - \frac{x^3}{5} + c$$

f $\frac{dy}{dx} - y \tan x = 1$

$$\Rightarrow \cos x \frac{dy}{dx} - y \sin x = \cos x$$

$$\Rightarrow \frac{d}{dx}\left(y \cos x\right) = \cos x$$

$$\Rightarrow y \cos x = \sin x + c$$

$$\Rightarrow y = \tan x + C \sec x$$

g $\frac{dy}{dx} - \frac{1}{2}y = \frac{1}{2}e^{\frac{1}{2}x}$

$$I = e^{\int -\frac{1}{2}dx} = e^{-\frac{x}{2}}$$

$$\therefore e^{-\frac{x}{2}}\frac{dy}{dx} - \frac{1}{2}e^{-\frac{x}{2}}y = \frac{1}{2}$$

$$\Rightarrow \frac{d}{dx}\left(e^{-\frac{1}{2}x}y\right) = \frac{1}{2}$$

$$\Rightarrow e^{-\frac{1}{2}x}y = \frac{x}{2} + c$$

$$\Rightarrow y = \left(\frac{x}{2} + c\right)e^{\frac{x}{2}} = \frac{1}{2}\left(x + c\right)e^{\frac{x}{2}}$$

5 a $\frac{dy}{dt} = ky$

b $\frac{1}{y}\frac{dy}{dt} = k$

$$\Rightarrow \ln y = kt + c$$

$\Rightarrow y = y_0 e^{kt}$ where y_0 is the amount of substance at $t = 0$

$$\frac{y_0}{2} = y_0 e^{5500k} \Rightarrow k = -\frac{\ln 2}{5500}$$

$$\therefore y = y_0 e^{-\frac{\ln 2}{5500}t}$$

c $\dfrac{y_0}{5} = y_0 \, e^{-\frac{\ln 2}{5500}t}$

$\Rightarrow \ln\left(\dfrac{1}{5}\right) = -\dfrac{\ln 2}{5500}t$

$\Rightarrow t = \dfrac{5500 \ln 5}{\ln 2} = 12770.60452$

so 13000 years (to the nearest thousand years)

6 a

	x_n	y_n	$\dfrac{dy}{dx} = f\left(x_n, y_n\right) = 1 - \dfrac{x_n y_n}{4 - x_n{}^2}$
0	0	1	1
1	0.25	1.25	0.9206349206
2	0.5	1.48015873	0.8026455026
3	0.75	1.680820106	0.6332756133
4	1	1.839139009	

$y(1) \approx 1.84$

b $\dfrac{dy}{dx} + \dfrac{x}{4 - x^2} y = 1$ subject to $y(0) = 1$

$I = e^{\int \frac{x}{4 - x^2} dx} = e^{-\frac{1}{2}\ln\left(4 - x^2\right)} = \dfrac{1}{\sqrt{4 - x^2}}$

$\therefore \dfrac{1}{\sqrt{4 - x^2}} \dfrac{dy}{dx} + \dfrac{x}{\left(4 - x^2\right)^{\frac{3}{2}}} y = \dfrac{1}{\sqrt{4 - x^2}}$

$\Rightarrow \dfrac{d}{dx}\left(\dfrac{1}{\sqrt{4 - x^2}} y\right) = \dfrac{1}{\sqrt{4 - x^2}}$

$\Rightarrow \dfrac{y}{\sqrt{4 - x^2}} = \arcsin\left(\dfrac{x}{2}\right) + c$

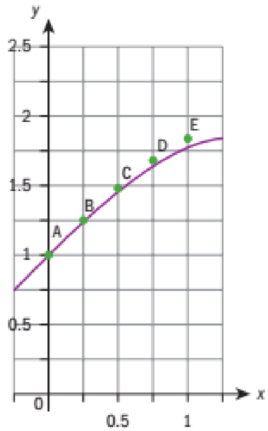
$\Rightarrow y = \sqrt{4 - x^2}\left(\arcsin\left(\dfrac{x}{2}\right) + c\right)$

$y(0) = 2c = 1 \Rightarrow c = \dfrac{1}{2}$

$\Rightarrow y = \sqrt{4 - x^2}\left(\arcsin\left(\dfrac{x}{2}\right) + \dfrac{1}{2}\right)$

$y(1) = 1.7729\ldots$

c



Since y' is decreasing the value of y is greater than the actual value.

7 $f(x) = \ln(1+\sin x) \Rightarrow f(0) = 0$

$$f'(x) = \frac{\cos x}{1+\sin x}$$

$\Rightarrow f'(0) = 1$

$$f''(x) = \frac{(1+\sin x)(-\sin x) - \cos x(\cos x)}{(1+\sin x)^2} = \frac{-1-\sin x}{(1+\sin x)^2} = -\frac{1}{1+\sin x}$$

$\Rightarrow f''(0) = -1$

$$f'''(0) = -\frac{-\cos x}{(1+\sin x)^2} = \frac{\cos x}{(1+\sin x)^2}$$

$\Rightarrow f'''(0) = 1$

$$f^{(4)}(x) = \frac{(1+\sin x)^2(-\sin x) - \cos x \bullet 2\cos x(1+\sin x)}{(1+\sin x)^4}$$

$$= \frac{-\sin x(1+\sin x) - 2\cos^2 x}{(1+\sin x)^3} = \frac{-\sin x - \sin^2 x - \cos^2 x - \cos^2 x}{(1+\sin x)^3}$$

$$= \frac{-\sin x - 1 - \cos^2 x}{(1+\sin x)^3}$$

$\therefore f^{(4)}(0) = -2$

$f^{(5)}(0) = 5$

$$\therefore f(x) = x - \frac{1}{2!}x^2 + \frac{1}{3!}x^3 - \frac{2}{4!}x^4 + \frac{5}{5!}x^5$$

$$= x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{24}$$

8 Using L'Hopital's Rule:

$$\lim_{x\to 0}\frac{\sin x - x}{x\sin x} = \lim_{x\to 0}\frac{\cos x - 1}{\sin x + x\cos x} = \lim_{x\to 0}\frac{-\sin x}{2\cos x - x\sin x} = 0$$

Using MacLaurin Series:

$$\lim_{x\to 0}\frac{\sin x - x}{x\sin x} = \lim_{x\to 0}\frac{\left(x-\frac{x^3}{6}+\frac{x^5}{120}+\ldots\right)-x}{x\left(x-\frac{x^3}{6}+\frac{x^5}{120}-\ldots\right)}$$

$$= \lim_{x\to 0}\frac{\left(-\frac{x^3}{6}+\frac{x^5}{120}-\ldots\right)}{x\left(x-\frac{x^3}{6}+\frac{x^5}{120}-\ldots\right)} = \frac{\lim_{x\to 0}\left(-\frac{x}{6}+\frac{x^3}{120}-\ldots\right)}{\lim_{x\to 0}\left(1-\frac{x^2}{6}+\ldots\right)} = 0$$

(dividing top and bottom by x^2 in penultimate step)

9 Using L'Hopital's Rule:

$$\lim_{x\to 1}\frac{\ln x}{\sin(2\pi x)} = \lim_{x\to 1}\frac{\frac{1}{x}}{2\pi\cos(2\pi x)} = \frac{1}{2\pi}$$

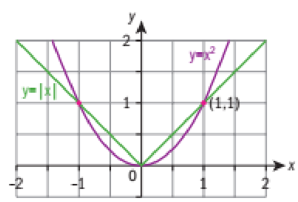
Using MacLaurin Series:

$$\lim_{x\to 1}\frac{\ln x}{\sin(2\pi x)} = \lim_{x\to 0}\frac{\ln(1+x)}{\sin(2\pi(x+1))} = \lim_{x\to 0}\frac{\ln(x+1)}{\sin(2\pi x)}$$

$$= \lim_{x\to 0}\frac{\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots\right)}{\left(2\pi x-\frac{(2\pi x)^3}{6}+\ldots\right)} = \lim_{x\to 0}\frac{\left(1-\frac{x}{2}+\frac{x^2}{3}+\ldots\right)}{2\pi-\frac{4}{3}\pi^3x^2+\ldots} = \frac{1}{2\pi}$$

Exam-style questions

10 a



(1 mark) for correct shape, (1 mark) for symmetry about the y-axis, (1 mark) for points of intersection

b A $= 2\int_0^1 (x - x^2)dx$ (1 mark)

$= 2\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1$ (1 mark)

$= 2\left(\frac{1}{2}-\frac{1}{3}\right)$ (1 mark)

$= \frac{1}{3}$ square units (1 mark)

11 a $d(t) = |\sin 2t - \sin(t-0.24)|$ (2 marks)

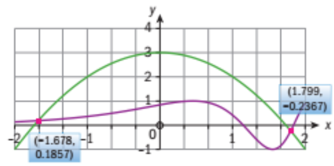
b Use GDC to find the maximum of $d = 1.88$ (1 mark)

occurs when $t = 2.25$ (1 mark)

c Find intersection of graphs (1 mark)

$t = 1.13$ (1 mark)

12 a



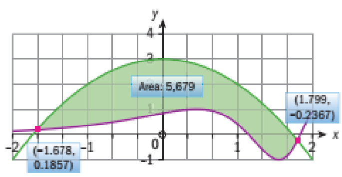
(1 mark) for shape, (1 mark) for domain

b $f(x) = g(x) \Rightarrow x = -1.68, x = 1.80$ (2 marks)

c $-1.68 \leq x \leq 1.80$ (1 mark)

d $\int_{-1.68}^{1.8} \left(f(x) - g(x)\right) dx = 5.68$ (2 marks)

OR can be done using technology



13 Let the number of insects be y.

$\frac{dy}{dx} = -ky$ (1 mark)

$\int \frac{1}{y} dy = \int -k dt$ (1 mark)

$\ln y = -kt + c$ (1 mark)

$y = e^{-kt+c}$

$y = Ae^{-kt}$

when $t = 0$, $y = 500\,000 \Rightarrow A = 500\,000$ (1 mark)

$y = 500\,000e^{-kt}$

when $t = 5$, $y = 400\,000$

$400\,000 = 500\,000e^{-5k}$ (1 mark)

$\frac{4}{5} = e^{-5k}$

$-5k = \ln\frac{4}{5}$

$k = -\frac{1}{5}\ln\frac{4}{5}$ $(= 0.0446)$ (1 mark)

$250\,000 = 500\,000e^{-kt}$ (1 mark)

$\frac{1}{2} = e^{-kt}$

$\ln\frac{1}{2} = -kt$

$t = \frac{5}{\ln\frac{4}{5}}\ln\frac{1}{2}$ $= 15.5$ years (1 mark)

14 a $\int \sec^2 y dy = \int \cos x dx$ (2 marks)

$\tan y = \sin x + c$ (1 mark)

$\tan\frac{\pi}{4} = \sin\pi + c \Rightarrow c = 1$ (1 mark)

tan y = 1 + sin x (1 mark)
y = arctan (1 + sin x)

b Since the denominator is 0 when $x = \frac{\pi}{2}$, to apply l'Hopital's rule the numerator must also be 0. (1 mark)

Hence $k = \arctan\left(1 + \sin\frac{\pi}{2}\right) = \arctan 2$ (1 mark)

$\lim_{x \to \frac{\pi}{2}} \frac{\arctan(1 + \sin x) - \arctan 2}{\left(x - \frac{\pi}{2}\right)^2} = \frac{0}{0}$, so by applying l'Hopital's rule:

$$\lim_{x \to \frac{\pi}{2}} \frac{\arctan(1 + \sin x) - \arctan 2}{\left(x - \frac{\pi}{2}\right)^2} = \lim_{x \to \frac{\pi}{2}} \frac{\frac{\cos x}{\sin^2 x + 2\sin x + 2}}{2\left(x - \frac{\pi}{2}\right)} = \frac{0}{0}$$ (2 marks)

Applying l'Hopital's rule again gives

$$= \lim_{x \to \frac{\pi}{2}} \frac{\frac{-(8 + 13\sin x + \sin 3x)}{(\cos 2x - 4\sin x - 5)^2}}{2} = \frac{\frac{-(8 + 13 - 1)}{(-1 - 4 - 5)^2}}{2} = \frac{\frac{-20}{100}}{2} = \frac{-1}{10}$$ (3 marks)

15 a y = ln (1 + sin x)

$y' = \frac{\cos x}{1 + \sin x}$ (1 mark)

$y'' = -\frac{1}{1 + \sin x}$ (1 mark)

$y^{(3)} = \frac{\cos x}{(1 + \sin x)^2}$ (1 mark)

$y^{(4)} = \frac{-\sin x(1 + \sin x)^2 - 2(1 + \sin x)\cos^2 x}{(1 + \sin x)^4}$ (2 marks)

$y(0) = 0$; $y'(0) = 1$ (1 mark)

$y''(0) = -1$; $y^{(3)}(0) = 1$; $y^{(4)}(0) = -2$ (2 marks)

$\ln(1 + \sin x) = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \ldots$

b i ln(1 − sin x) = ln(1 + sin(−x)) (1 mark)

$= -x - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{12}x^4 + \ldots$ (1 mark)

ii ln(1 + sin x) + ln(1 − sin x) = ln(1 − sin^2 x) (1 mark)
= ln cos^2 x (1 mark)

$\ln\cos^2 x = -x^2 - \frac{1}{6}x^4 + \ldots$ (1 mark)

$\ln\cos x = \frac{1}{2}\ln\cos^2 x = -\frac{1}{2}x^2 - \frac{1}{12}x^4 + \ldots$ (1 mark)

iii $\dfrac{d}{dx}(\ln\cos x) = \dfrac{1}{\cos x} \times (-\sin x)$ (1 mark)

$= -\tan x$ (1 mark)

$\tan x = x + \dfrac{1}{3}x^3 + \ldots$ (2 marks)

c $\dfrac{\tan\left(x^2\right)}{\ln\cos x} = \dfrac{x^2 + \dfrac{x^4}{3} + \ldots}{-\dfrac{x^2}{2} - \dfrac{x^4}{12} + \ldots}$ (1 mark)

$= \dfrac{1 + \dfrac{x^2}{3} + \ldots}{-\dfrac{1}{2} - \dfrac{x^2}{12} + \ldots}$ (1 mark)

$\to -2$ as $x \to 0$ (1 mark)

$\lim\limits_{x\to\infty}\left(\dfrac{\tan\left(x^2\right)}{\ln\cos x}\right) = -2$ (1 mark)

9 Modelling 3D space: vectors

Skills check

1 a $c = \sqrt{21^2 + 20^2} = 29$

b $a = \sqrt{13^2 - 7^2} = 10.95$

2 $d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2} = \sqrt{(-5-3)^2 + (4+2)^2} = 10$

3 We calculate the slopes and get that $m_a = -\frac{2}{3} = m_c$ so a and c are parallel

$m_b = \frac{3}{2}$, $m_a \times m_b = m_a \times m_d = -1$ so b and d are parallel, and both perpendicular to a. e is not parallel or perpendicular to any of the other lines.

4 We use method of elimination

$$\begin{cases} 16x - 12y = 4 \\ -15x + 12y = -6 \end{cases}$$

We subtract both and get that $x = -2$

and so $4(-2) - 3y = 1$

$y = \frac{1+8}{-3} = -3$

Exercise 9A

1 correct vectors drawn

2 a $\boldsymbol{a} + \boldsymbol{b}$ **b** $-\boldsymbol{a} - \boldsymbol{b}$ **c** $\boldsymbol{b} - \boldsymbol{a}$ **d** $-2\boldsymbol{a} - \boldsymbol{b}$

e $-2\boldsymbol{a} - \frac{3}{2}\boldsymbol{b}$ **f** $2\boldsymbol{b} + \frac{3}{2}\boldsymbol{a}$ **g** $-\frac{3\boldsymbol{a}}{2} + \frac{\boldsymbol{b}}{2}$ **h** $\frac{1}{2}\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}$

3 a AG = AB + BC +CG = **a** + **b** + **c**

b CE = CB + BA + AE = −**b** − **a** + **c**

c DF = DA + AB + BF = −**b** + **a** + **c**

d MN = MB + BC + CG + GN = $-\frac{1}{2}\boldsymbol{c} + \boldsymbol{b} + \boldsymbol{c} - \frac{1}{2}\boldsymbol{a} = \boldsymbol{b} + \frac{1}{2}\boldsymbol{c} - \frac{1}{2}\boldsymbol{a}$

4 i $\lambda(\mu\boldsymbol{a}) = \lambda\mu\boldsymbol{a} = (\lambda\mu)\boldsymbol{a} = (\mu\lambda)\boldsymbol{a} = \mu(\lambda\boldsymbol{a})$

ii By distributivity of scalar multiplication, $\lambda(\boldsymbol{a} + \boldsymbol{b}) = \lambda\boldsymbol{a} + \lambda\boldsymbol{b}$

iii By distributivity of scalar multiplication, $(\lambda + \mu)\boldsymbol{a} = \lambda\boldsymbol{a} + \mu\boldsymbol{a}$

iv 1 is the identity so the operation from left or right returns the same vector, $1 \cdot \boldsymbol{a} = \boldsymbol{a} \cdot 1 = \boldsymbol{a}$

v Multiplying by zero will always return the zero vector, $0 \cdot \boldsymbol{a} = \boldsymbol{0}$

Exercise 9B

1 We need to show that

$BD = \lambda PQ$

so we use the triangle rule for both diagonals to get

$BC + CD = BD$

and

$PC + CQ = PQ$

where

$PC = \frac{1}{2} BC$

and

$CQ = \frac{1}{2} CD$

Then

$BD = 2(PC + CQ) = 2PQ$

hence they are parallel. Additionally,

$PQ = \frac{1}{2} BD$

2 If PQ is perpendicular to AC, then PQ is parallel to BD, as the diagonals are perpendicular. Then

$PQ = QC + CP$

$AC = AD + DC$

$BD = DC + CB$

$BD = 2QC + 2CP = 2PQ \Rightarrow BD$ and PQ are parallel, hence PQ is orthogonal to AC

$AD = 2CP$, $DC = 2QC$

Then

$AC = 2CP + 2QC = 2(CP + QC) = 2PQ$

as requested

3 On the side $[AB]$ draw the point E such that $\boldsymbol{BC} = \boldsymbol{ED}$. Hence

$$\boldsymbol{NM} = \boldsymbol{ND} + \boldsymbol{DC} + \boldsymbol{CM} = \frac{1}{2}\boldsymbol{AB} + \boldsymbol{DC} + \frac{1}{2}\boldsymbol{CB}$$

$$= \frac{1}{2}\boldsymbol{AB} + \frac{1}{2}\boldsymbol{DE} + \boldsymbol{DC} = \frac{1}{2}\boldsymbol{AE} + \frac{1}{2}\boldsymbol{DC} + \frac{1}{2}\boldsymbol{DC}$$

$$= \frac{1}{2}\boldsymbol{AE} + \frac{1}{2}\boldsymbol{EB} + \frac{1}{2}\boldsymbol{DC} = \frac{1}{2}\boldsymbol{AB} + \frac{1}{2}\boldsymbol{DC}$$

We know that $[AB]$ and $[DC]$ are parallel, hence since $[NM]$ is a linear combination of them then all three of them are mutually parallel and the same equation is true for the magnitudes.

$$|\boldsymbol{NM}| = \frac{1}{2}|\boldsymbol{AB}| + \frac{1}{2}|\boldsymbol{DC}|$$

4 We have that $DC = \frac{5}{3}y$ where y is some length. Then the ratio gives us that $\frac{3}{5}PC = y$.

5 **a** $HC = HF + FE + EC$ where

$FE = AB$

$HC - HQ = AB$

$HC - \lambda HC = AB$

$HC = (1-k)AB$

Hence they are parallel.

This means we can form a right angled triangle HQA and Pythagoras' theorem gives us

$HC = \left(1+\sqrt{2}\right)AB$

b $MN = ME + ED + DN$

$MN = \frac{1}{2}FE + ED + \frac{1}{2}DC$

$MN = AD + \frac{1}{2}OD + \frac{1}{2}DC$

Hence they are parallel.

Again we have a right angled triangle which gives us that

$MN = \left(1+\frac{\sqrt{2}}{2}\right)AB$

6 $KL = KB + BL$

$NM = ND + DM$

We know that they are the midpoints, so

$KL = \frac{AB}{2} + \frac{BC}{2}$

and

$NM = \frac{AD}{2} + \frac{DC}{2}$

Then we form a parallelogram.

Exercise 9C

1 **a** $\boldsymbol{a} + \boldsymbol{b} = (2-3)\boldsymbol{i} + (-5+4)\boldsymbol{j} = -\boldsymbol{i} - \boldsymbol{j}$

b $\boldsymbol{a}-\boldsymbol{b}=(2+3)\boldsymbol{i}+(-5-4)\boldsymbol{j}=5\boldsymbol{i}-9\boldsymbol{j}$

c $5\boldsymbol{a}-6\boldsymbol{b}=5(2\boldsymbol{i}-5\boldsymbol{j})-6(-3\boldsymbol{i}+4\boldsymbol{j})=10\boldsymbol{i}-25\boldsymbol{j}+18\boldsymbol{i}-24\boldsymbol{j}=28\boldsymbol{i}-49\boldsymbol{j}$

d $7\boldsymbol{b}-4\boldsymbol{a}=7(-3\boldsymbol{i}+4\boldsymbol{j})-4(2\boldsymbol{i}-5\boldsymbol{j})=-21\boldsymbol{i}+28\boldsymbol{j}-8\boldsymbol{i}+20\boldsymbol{j}=-29\boldsymbol{i}+48\boldsymbol{j}$

e $\frac{3}{5}\boldsymbol{a}+\frac{3}{4}\boldsymbol{b}=\frac{3}{5}(2\boldsymbol{i}-5\boldsymbol{j})+\frac{3}{4}(-3\boldsymbol{i}+4\boldsymbol{j})=\frac{6}{5}\boldsymbol{i}-\frac{15}{5}\boldsymbol{j}-\frac{9}{4}\boldsymbol{i}+\frac{12}{4}\boldsymbol{j}=-\frac{21}{20}\boldsymbol{i}$

2 **a** Let $\alpha(3\boldsymbol{i}+2\boldsymbol{j})+\beta(\boldsymbol{i}+5\boldsymbol{j})=5\boldsymbol{i}-\boldsymbol{j}$

where α and β are constants. Then

$3\alpha\boldsymbol{i}+2\alpha\boldsymbol{j}+\beta\boldsymbol{i}+5\beta\boldsymbol{j}=5\boldsymbol{i}-\boldsymbol{j}$

This gives two equations

$3\alpha+\beta=5$

$2\alpha+5\beta=-1$

Then

$\beta=5-3\alpha$

we substitute to get α

$2\alpha+5(5-3\alpha)=-1$

$-13\alpha=-26$

$\alpha=2$

and we substitute again to get β

$\beta=5-3(2)=-1$

$5\mathbf{i}-\mathbf{j}=2\mathbf{p}-\mathbf{q}$

b Let

$\alpha(3\boldsymbol{i}+2\boldsymbol{j})+\beta(\boldsymbol{i}+5\boldsymbol{j})=10\boldsymbol{i}+9\boldsymbol{j}$

where α and β are constants. Then

$3\alpha\boldsymbol{i}+2\alpha\boldsymbol{j}+\beta\boldsymbol{i}+5\beta\boldsymbol{j}=10\boldsymbol{i}-9\boldsymbol{j}$

This gives two equations

$3\alpha+\beta=10$

$2\alpha+5\beta=9$

Then

$\beta=10-3\alpha$

we substitute to get α.

$2\alpha+5(10-3\alpha)=9$

$-13\alpha = -41$

$\alpha = \frac{41}{13}$

and we substitute again to get β

$\beta = 10 - 3\left(\frac{41}{13}\right) = \frac{7}{13}$

$\frac{41}{13}\boldsymbol{p} + \frac{7}{13}\boldsymbol{q}$

c Let

$\alpha(3\boldsymbol{i} + 2\boldsymbol{j}) + \beta(\boldsymbol{i} + 5\boldsymbol{j}) = -9\boldsymbol{i} + 7\boldsymbol{j}$

where α and β are constants. Then

$3\alpha\boldsymbol{i} + 2\alpha\boldsymbol{j} + \beta\boldsymbol{i} + 5\beta\boldsymbol{j} = -9\boldsymbol{i} + 7\boldsymbol{j}$

This gives two equations

$3\alpha + \beta = -9$

$2\alpha + 5\beta = 7$

Then

$\beta = -9 - 3\alpha$

we substitute to get α

$2\alpha + 5(-9 - 3\alpha) = 7$

$-13\alpha = 52$

$\alpha = -4$

and we substitute again to get β

$\beta = -9 - 3(-4) = 3$

$-4\mathbf{p} + 3\mathbf{q}$

d Let

$\alpha(3\boldsymbol{i} + 2\boldsymbol{j}) + \beta(\boldsymbol{i} + 5\boldsymbol{j}) = \boldsymbol{i}$

where α and β are constants. Then

$3\alpha\boldsymbol{i} + 2\alpha\boldsymbol{j} + \beta\boldsymbol{i} + 5\beta\boldsymbol{j} = \boldsymbol{i}$

This gives two equations

$3\alpha + \beta = 1$

$2\alpha + 5\beta = 0$

Then

$\beta = 1 - 3\alpha$

we substitute to get α

$$2\alpha + 5(1 - 3\alpha) = 0$$

$$-13\alpha = -5$$

$$\alpha = \frac{5}{13}$$

and we substitute again to get β

$$\beta = 1 - 3\left(\frac{5}{13}\right) = \frac{-2}{13}$$

$$\frac{5}{13}\boldsymbol{p} - \frac{2}{13}\boldsymbol{q}$$

e Let

$$\alpha(3\boldsymbol{i} + 2\boldsymbol{j}) + \beta(\boldsymbol{i} + 5\boldsymbol{j}) = -\boldsymbol{j}$$

where α and β are constants. Then

$$3\alpha\boldsymbol{i} + 2\alpha\boldsymbol{j} + \beta\boldsymbol{i} + 5\beta\boldsymbol{j} = -\boldsymbol{j}$$

This gives two equations

$$3\alpha + \beta = 0$$

$$2\alpha + 5\beta = -1$$

Then

$$\beta = -3\alpha$$

we substitute to get α

$$2\alpha + 5(-3\alpha) = -1$$

$$-13\alpha = -1$$

$$\alpha = \frac{1}{13}$$

and we substitute again to get β

$$\beta = -3\left(\frac{1}{13}\right) = \frac{-3}{13}$$

$$\frac{1}{13}\boldsymbol{p} - \frac{3}{13}\boldsymbol{q}$$

f Let

$$\alpha(3\boldsymbol{i} + 2\boldsymbol{j}) + \beta(\boldsymbol{i} + 5\boldsymbol{j}) = -\frac{1}{2}\boldsymbol{i} + \frac{2}{3}\boldsymbol{j}$$

where α and β are constants. Then

$$3\alpha\boldsymbol{i} + 2\alpha\boldsymbol{j} + \beta\boldsymbol{i} + 5\beta\boldsymbol{j} = -\frac{1}{2}\boldsymbol{i} + \frac{2}{3}\boldsymbol{j}$$

This gives two equations

$$3\alpha + \beta = -\frac{1}{2}$$

$$2\alpha + 5\beta = \frac{2}{3}$$

Then

$$\beta = -\frac{1}{2} - 3\alpha$$

we substitute to get α

$$2\alpha + 5\left(-\frac{1}{2} - 3\alpha\right) = \frac{2}{3}$$

$$-13\alpha = \frac{19}{6}$$

$$\alpha = \frac{-19}{78}$$

and we substitute again to get β

$$\beta = -\frac{1}{2} - 3\left(\frac{-19}{78}\right) = \frac{3}{13}$$

$$-\frac{19}{78}\boldsymbol{p} + \frac{3}{13}\boldsymbol{q}$$

3 Note that

$$QR \equiv PS$$

so we calculate

$$QR = -i - 3j + 4i + j = 3i - 2j$$

and so

$$PS = (3 - x)i + (3 - y)j$$

where $P = xi + yj$, corresponding to (x, y) coordinates of P. Then we equate both expressions and get

$$3 - x = 3$$

$$3 - y = -2$$

so $x = 0$ and $y = 5$. Then $P = (0, 5)$

4 In the notation bellow, any vector with a single letter is measured from the origin (e.g. OA=A)

$$OA = OB + CD = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} i \\ j \\ k \end{pmatrix} + \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} i \\ j \\ k \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} i \\ j \\ k \end{pmatrix}$$

$$OE = OA + CG = \begin{pmatrix}2\\0\\1\end{pmatrix}\cdot\begin{pmatrix}i\\j\\k\end{pmatrix} + \begin{pmatrix}-2\\0\\-2\end{pmatrix}\cdot\begin{pmatrix}i\\j\\k\end{pmatrix} = \begin{pmatrix}0\\0\\-1\end{pmatrix}\begin{pmatrix}i\\j\\k\end{pmatrix}$$

$$OF = OB + CG = \begin{pmatrix}3\\2\\0\end{pmatrix}\cdot\begin{pmatrix}i\\j\\k\end{pmatrix} + \begin{pmatrix}-2\\0\\-2\end{pmatrix}\cdot\begin{pmatrix}i\\j\\k\end{pmatrix} = \begin{pmatrix}1\\2\\-2\end{pmatrix}\begin{pmatrix}i\\j\\k\end{pmatrix}$$

$$OH = OD + CG = \begin{pmatrix}1\\1\\2\end{pmatrix}\cdot\begin{pmatrix}i\\j\\k\end{pmatrix} + \begin{pmatrix}-2\\0\\-2\end{pmatrix}\cdot\begin{pmatrix}i\\j\\k\end{pmatrix} = \begin{pmatrix}-1\\1\\0\end{pmatrix}\begin{pmatrix}i\\j\\k\end{pmatrix}$$

Then $A = (2,0,1)$, $E = (0,0,-1)$, $F = (1,2,-2)$, $H = (-1,1,0)$

5 **i** Commutative: Let $\boldsymbol{a} = x\boldsymbol{i} + y\boldsymbol{j}$, and $\boldsymbol{b} = m\boldsymbol{i} + n\boldsymbol{j}$ for real x,y,m,n . Then for all $\boldsymbol{a},\boldsymbol{b}$

$$\boldsymbol{a} + \boldsymbol{b} = (x+m)\boldsymbol{i} + (y+n)\boldsymbol{j} = (m+x)\boldsymbol{i} + (n+y)\boldsymbol{j} = \boldsymbol{b} + \boldsymbol{a}$$

where we have used the commutativity of addition of real numbers.

ii Associative: Let $\boldsymbol{c} = l\boldsymbol{i} + p\boldsymbol{j}$ for real l,p, then for all $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$

$$(\boldsymbol{a}+\boldsymbol{b}) + \boldsymbol{c} = (m+x)\boldsymbol{i} + (n+y)\boldsymbol{j} + l\boldsymbol{i} + p\boldsymbol{j} = (m+x+l)\boldsymbol{i} + (n+y+p)\boldsymbol{j}$$

$$= x\boldsymbol{i} + y\boldsymbol{j} + (m+l)\boldsymbol{i} + (n+p)\boldsymbol{j} = \boldsymbol{a} + (\boldsymbol{b}+\boldsymbol{c})$$

where we have used the commutativity and associativity of addition of real numbers.

iii Identity: for $0 = 0\boldsymbol{i} + 0\boldsymbol{j}$ and for all $\boldsymbol{a}$

$$0 + \boldsymbol{a} = (0+x)\boldsymbol{i} + (0+y)\boldsymbol{j} = (x+0)\boldsymbol{i} + (y+0)\boldsymbol{j} = x\boldsymbol{i} + y\boldsymbol{j} = \boldsymbol{a}$$

where we have used the identity and commutativity of addition of real numbers.

iv Let $-\boldsymbol{a} = -x\boldsymbol{i} - y\boldsymbol{j}$. Then for all $\boldsymbol{a},-\boldsymbol{a}$

$$\boldsymbol{a} + (-\boldsymbol{a}) = (x-x)\boldsymbol{i} + (y-y)\boldsymbol{j} = (-x+x)\boldsymbol{i} + (-y+y)\boldsymbol{j} = -\boldsymbol{a} + \boldsymbol{a} = 0\boldsymbol{i} + 0\boldsymbol{j} = 0$$

where we have used the identity and commutativity of addition of real numbers.

6 **i** For any real λ, μ and for all $\boldsymbol{a} = x\boldsymbol{i} + y\boldsymbol{j}$ for real x,y we have

$$(\lambda\mu)\boldsymbol{a} = (\lambda\mu)(x\boldsymbol{i} + y\boldsymbol{j}) = \lambda\mu x\boldsymbol{i} + \lambda\mu y\boldsymbol{j} = \lambda(\mu x\boldsymbol{i}) + \lambda(\mu y\boldsymbol{j}) = \lambda(\mu\boldsymbol{a})$$

and

$$\lambda(\mu x\boldsymbol{i}) + \lambda(\mu y\boldsymbol{j}) = \mu(\lambda x\boldsymbol{i}) + \mu(\lambda y\boldsymbol{j}) = \mu(\lambda\boldsymbol{a})$$

where we have used the commutativity of the multiplication of real numbers.

ii Let $b = m\boldsymbol{i} + n\boldsymbol{j}$ for any real n,m

$$\lambda(\boldsymbol{a}+\boldsymbol{b}) = \lambda((x+m)\boldsymbol{i} + (y+n)\boldsymbol{j}) = \lambda x\boldsymbol{i} + \lambda m\boldsymbol{i} + \lambda y\boldsymbol{j} + \lambda n\boldsymbol{j} = \lambda x\boldsymbol{i} + \lambda y\boldsymbol{j} + \lambda m\boldsymbol{i} + \lambda n\boldsymbol{j} = \lambda\boldsymbol{a} + \lambda\boldsymbol{b}$$

where we have used the commutativity and associativity of the multiplication of real numbers.

iii $(\lambda+\mu)\boldsymbol{a}=(\lambda+\mu)(x\boldsymbol{i}+y\boldsymbol{j})=\lambda x\boldsymbol{i}+\mu x\boldsymbol{i}+\lambda y\boldsymbol{j}+\mu y\boldsymbol{j}=\lambda x\boldsymbol{i}+\lambda y\boldsymbol{j}+\mu x\boldsymbol{i}+\mu y\boldsymbol{j}=\lambda\boldsymbol{a}+\mu\boldsymbol{b}$

where we have used the commutativity and associativity of the multiplication of real numbers.

iv $1\boldsymbol{a}=1(x\boldsymbol{i}+y\boldsymbol{j})=(1\times x)\boldsymbol{i}+(1\times y)\boldsymbol{j}=x\boldsymbol{i}+y\boldsymbol{j}=\boldsymbol{a}$

where we have used the identity of multiplication of real numbers

v $0\boldsymbol{a}=0(x\boldsymbol{i}+y\boldsymbol{j})=(0\times x)\boldsymbol{i}+(0\times y)\boldsymbol{j}=0\boldsymbol{i}+0\boldsymbol{j}=0$

and

$\lambda(0\boldsymbol{i}+0\boldsymbol{j})=(\lambda\times 0)\boldsymbol{i}+(\lambda\times 0)\boldsymbol{j}=0\boldsymbol{i}+0\boldsymbol{j}=0$

Exercise 9D

1 a $\hat{\boldsymbol{a}}=\dfrac{7\boldsymbol{i}+24\boldsymbol{j}}{\sqrt{7^2+24^2}}=\dfrac{7\boldsymbol{i}+24\boldsymbol{j}}{25}=\dfrac{7}{25}\boldsymbol{i}+\dfrac{24}{25}\boldsymbol{j}$

b $\hat{\boldsymbol{b}}=\dfrac{-3\boldsymbol{i}+2\boldsymbol{j}}{\sqrt{3^2+2^2}}=\dfrac{-3\boldsymbol{i}+2\boldsymbol{j}}{\sqrt{13}}=\dfrac{-3}{\sqrt{13}}\boldsymbol{i}+\dfrac{2}{\sqrt{13}}\boldsymbol{j}$

c $\hat{\boldsymbol{c}}=\dfrac{4\boldsymbol{i}-5\boldsymbol{j}+20\boldsymbol{k}}{\sqrt{4^2+5^2+20^2}}=\dfrac{4\boldsymbol{i}-5\boldsymbol{j}+20\boldsymbol{k}}{21}=\dfrac{4}{21}\boldsymbol{i}-\dfrac{5}{21}\boldsymbol{j}+\dfrac{20}{21}\boldsymbol{k}$

d $\hat{\boldsymbol{d}}=\dfrac{-\boldsymbol{i}+3\boldsymbol{j}+4\boldsymbol{k}}{\sqrt{1^2+3^2+4^2}}=\dfrac{-\boldsymbol{i}+3\boldsymbol{j}+4\boldsymbol{k}}{\sqrt{26}}=-\dfrac{1}{\sqrt{26}}\boldsymbol{i}+\dfrac{3}{\sqrt{26}}\boldsymbol{j}+\dfrac{4}{\sqrt{26}}\boldsymbol{k}$

2 a $\hat{\boldsymbol{a}}=\dfrac{20\boldsymbol{i}-21\boldsymbol{j}}{\sqrt{20^2+21^2}}=\dfrac{20\boldsymbol{i}-21\boldsymbol{j}}{29}=\dfrac{20}{29}\boldsymbol{i}-\dfrac{21}{29}\boldsymbol{j}$

All vectors parallel to $\hat{\boldsymbol{a}}$ are of the form $\lambda\hat{\boldsymbol{a}}$ for real λ

b $\hat{\boldsymbol{b}}=\dfrac{\boldsymbol{i}-3\boldsymbol{j}}{\sqrt{1^2+3^2}}=\dfrac{\boldsymbol{i}-3\boldsymbol{j}}{\sqrt{10}}=\dfrac{1}{\sqrt{10}}\boldsymbol{i}-\dfrac{3}{\sqrt{10}}\boldsymbol{j}$

All vectors parallel to $\hat{\boldsymbol{b}}$ are of the form $\lambda\hat{\boldsymbol{b}}$ for real λ

c $\hat{\boldsymbol{c}}=\dfrac{5\boldsymbol{i}+6\boldsymbol{j}-30\boldsymbol{k}}{\sqrt{5^2+6^2+30^2}}=\dfrac{5\boldsymbol{i}+6\boldsymbol{j}-30\boldsymbol{k}}{31}=\dfrac{5}{31}\boldsymbol{i}+\dfrac{6}{31}\boldsymbol{j}-\dfrac{30}{31}\boldsymbol{k}$

All vectors parallel to $\hat{\boldsymbol{c}}$ are of the form $\lambda\hat{\boldsymbol{c}}$ for real λ

d $\hat{\boldsymbol{d}}=\dfrac{2\boldsymbol{i}+\boldsymbol{j}-5\boldsymbol{k}}{\sqrt{2^2+1^2+5^2}}=\dfrac{2\boldsymbol{i}+\boldsymbol{j}-5\boldsymbol{k}}{\sqrt{30}}=\dfrac{2}{\sqrt{30}}\boldsymbol{i}+\dfrac{1}{\sqrt{30}}\boldsymbol{j}-\dfrac{5}{\sqrt{30}}\boldsymbol{k}$

All vectors parallel to $\hat{\boldsymbol{d}}$ are of the form $\lambda\hat{\boldsymbol{d}}$ for real λ

3 $2|\boldsymbol{a}|=|\boldsymbol{b}|$

$$4|a|^2 = |b|^2$$
$$4\left(3^2+2^2+\lambda^2\right) = 1^2+5^2+(\lambda-5)^2$$
$$3\lambda^2+10\lambda+1=0$$
$$\lambda = \frac{-10 \pm \sqrt{100-12}}{6} = \frac{-5 \pm \sqrt{22}}{3}$$

4 **a** $\hat{\boldsymbol{a}} = \dfrac{5\boldsymbol{i}-\boldsymbol{j}}{\sqrt{5^2+1^2}} = \dfrac{5\boldsymbol{i}-\boldsymbol{j}}{\sqrt{26}} = \dfrac{5}{\sqrt{26}}\boldsymbol{i} - \dfrac{1}{\sqrt{26}}\boldsymbol{j}$

Then the required vector is $m\hat{\boldsymbol{a}} = \dfrac{5\times 6}{\sqrt{26}}\boldsymbol{i} - \dfrac{6}{\sqrt{26}}\boldsymbol{j} = \dfrac{30}{\sqrt{26}}\boldsymbol{i} - \dfrac{6}{\sqrt{26}}\boldsymbol{j}$

b $\hat{\boldsymbol{b}} = \dfrac{-4\boldsymbol{i}+5\boldsymbol{j}+20\boldsymbol{k}}{\sqrt{4^2+5^2+20^2}} = \dfrac{-4\boldsymbol{i}+5\boldsymbol{j}+20\boldsymbol{k}}{21} = \dfrac{-4}{21}\boldsymbol{i} + \dfrac{5}{21}\boldsymbol{j} + \dfrac{20}{21}\boldsymbol{k}$

Then the required vector is $m\hat{\boldsymbol{b}} = \dfrac{-4\times 63}{21}\boldsymbol{i} + \dfrac{5\times 63}{21}\boldsymbol{j} + \dfrac{20\times 63}{21}\boldsymbol{k} = -12\boldsymbol{i}+15\boldsymbol{j}+60\boldsymbol{k}$

5 **a** This is the same cuboid as in exercise 9C, 4. A space diagonal could be

$$\boldsymbol{AG} = \left[\begin{pmatrix}0\\3\\-1\end{pmatrix} - \begin{pmatrix}2\\0\\1\end{pmatrix}\right]\cdot\begin{pmatrix}i\\j\\k\end{pmatrix} = -2\boldsymbol{i}+3\boldsymbol{j}-2\boldsymbol{k}$$
$$\Rightarrow |\boldsymbol{AG}| = \sqrt{17}$$

b Recall that $A=(2,0,1)$, $E=(0,0,-1)$, $F=(1,2,-2)$, $H=(-1,1,0)$

Then

$$AD = -\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$$

$$AE = -2\boldsymbol{i}+0\boldsymbol{j}-2\boldsymbol{k}$$

$$AB = \boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}$$

$$V = A_{base}\times h = |AD||AE||AB| = \sqrt{1^2+1^2+1^2}\sqrt{2^2+2^2}\sqrt{1^2+2^2+1^2} = 12$$

Exercise 9E

1 **a** $\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta = \sqrt{3}\times 4\cos 30^\circ = 6$

b $\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta = 12\times 8\cos 115^\circ = -40.6$

c $\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta = 3\times 5\cos\dfrac{\pi}{7} = 13.5$

d $\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta = 5\sqrt{2}\times 17\cos\dfrac{3\pi}{4} = -85$

2 **a** $\boldsymbol{a}\cdot\boldsymbol{b} = 3\cdot 6+(-4)\cdot 5 = -2$

b $|\boldsymbol{a}| = \sqrt{3^2+4^2} = 5$

$|\boldsymbol{b}| = \sqrt{6^2 + 5^2} = \sqrt{61}$

$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta = 5\sqrt{61} \cos\theta = -2$

$\cos\theta = \dfrac{-2}{5\sqrt{61}}$

$\theta = 1.62\text{rad} = 93°$

3 **a** $\boldsymbol{a} \cdot \boldsymbol{b} = 1 \cdot (-2) + 4 \cdot 3 + (-3) \cdot 1 = 7$

b $|\boldsymbol{a}| = \sqrt{1^2 + 4^2 + 3^2} = \sqrt{26}$

$|\boldsymbol{b}| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$

$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta = \sqrt{26 \times 14} \cos\theta = 7$

$\cos\theta = \dfrac{7}{2\sqrt{91}}$

$\theta = 1.20\text{rad} = 68.5°$

4 $(\boldsymbol{a} - 2\boldsymbol{b}) \cdot (2\boldsymbol{a} + \boldsymbol{b}) = 2\boldsymbol{a} \cdot \boldsymbol{a} + \boldsymbol{a} \cdot \boldsymbol{b} - 2\boldsymbol{b} \cdot 2\boldsymbol{a} - 2\boldsymbol{b} \cdot \boldsymbol{b} = 0$

$2 \times 2 \times 2 + \boldsymbol{a} \cdot \boldsymbol{b} - 4(\boldsymbol{a} \cdot \boldsymbol{b}) - 2\sqrt{3}\sqrt{3} = 0$

$\boldsymbol{a} \cdot \boldsymbol{b} = \dfrac{2}{3} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta = 2\sqrt{3} \cos\theta$

$\theta = \cos^{-1} \dfrac{1}{3\sqrt{3}}$

$\theta = 1.38\text{rad} = 78.9°$

5 Let $\boldsymbol{a} = x\boldsymbol{i} + y\boldsymbol{j}$ and $\boldsymbol{b} = m\boldsymbol{i} + n\boldsymbol{j}$ for any real x, y, m, n. Then

i $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta = |\boldsymbol{b}||\boldsymbol{a}| \cos\theta = \boldsymbol{b} \cdot \boldsymbol{a}$

ii $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}||\boldsymbol{a}| \cos 0 = |\boldsymbol{a}|^2$

We prove it for the two dimensional case.

iii Let $\boldsymbol{a} = x\boldsymbol{i} + y\boldsymbol{j}$, $\boldsymbol{b} = m\boldsymbol{i} + n\boldsymbol{j}$, and $\boldsymbol{c} = s\boldsymbol{i} + t\boldsymbol{j}$ for any real x, y, m, n. Then

$\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = x \times (m + s) + y \times (n + t) = xm + xs + yn + yt = xm + yn + xs + yt = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$

We have used multiplicative properties for real numbers, therefore this can be extended to any dimension of vector, as the associativity and distributivity of scalar multiplication holds.

iv Let $\lambda \in \mathrm{R} \Rightarrow \lambda(\boldsymbol{a} \cdot \boldsymbol{b}) = \lambda(|\boldsymbol{a}||\boldsymbol{b}| \cos\theta) = \lambda|\boldsymbol{a}||\boldsymbol{b}| \cos\theta = (\lambda|\boldsymbol{a}|)|\boldsymbol{b}| \cos\theta = |\boldsymbol{a}|(\lambda|\boldsymbol{b}|) \cos\theta$

Hence

$\lambda(\boldsymbol{a} \cdot \boldsymbol{b}) = (\lambda\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (\lambda\boldsymbol{b})$.

6 **i** $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{a} + 2\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{b} = |\boldsymbol{a}|^2 + 2|\boldsymbol{a}||\boldsymbol{b}| + |\boldsymbol{b}|^2 = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + 2|\boldsymbol{a}||\boldsymbol{b}| \cos\theta$

ii $(\boldsymbol{a}-\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b}) = \boldsymbol{a}\cdot\boldsymbol{a} - 2\boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{b}\cdot\boldsymbol{b} = |\boldsymbol{a}|^2 - 2|\boldsymbol{a}||\boldsymbol{b}| + |\boldsymbol{b}|^2 = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2|\boldsymbol{a}||\boldsymbol{b}|\cos\theta$

Each of these cases correspond to the cosine rule for a triangle with sides $\boldsymbol{a}, \boldsymbol{b},$ and $\boldsymbol{a}\pm\boldsymbol{b}$.

7 We use both definitions to write the scalar product between $\boldsymbol{a}$ and $\boldsymbol{b}$, i.e.

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_1b_1 + a_2b_2 = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$$

Hence both definitions are equivalent.

8 We form the systems of equations

$$(\boldsymbol{a}-\boldsymbol{b})\cdot(2\boldsymbol{a}+\boldsymbol{b}) = \boldsymbol{a}\cdot 2\boldsymbol{a} + \boldsymbol{a}\cdot\boldsymbol{b} - \boldsymbol{b}\cdot 2\boldsymbol{a} - \boldsymbol{b}\cdot\boldsymbol{b} = 0$$

and

$$(\boldsymbol{a}-2\boldsymbol{b})\cdot(3\boldsymbol{a}+\boldsymbol{b}) = \boldsymbol{a}\cdot 3\boldsymbol{a} + \boldsymbol{a}\cdot\boldsymbol{b} - 2\boldsymbol{b}\cdot 3\boldsymbol{a} - 2\boldsymbol{b}\cdot\boldsymbol{b} = 0$$

This simplifies to

$$2(\boldsymbol{a}\cdot\boldsymbol{a}) - \boldsymbol{b}\cdot\boldsymbol{b} - \boldsymbol{a}\cdot\boldsymbol{b} = 0$$

and

$$3(\boldsymbol{a}\cdot\boldsymbol{a}) - 2\boldsymbol{b}\cdot\boldsymbol{b} - 5(\boldsymbol{a}\cdot\boldsymbol{b}) = 0$$

We will express $|\boldsymbol{a}|^2$ and $|\boldsymbol{b}|^2$ in terms of the scalar product between a and b. This means we solve the system of equations for the norms of a and b

$$\boldsymbol{b}\cdot\boldsymbol{b} = 2(\boldsymbol{a}\cdot\boldsymbol{a}) - \boldsymbol{a}\cdot\boldsymbol{b}$$

$$3(\boldsymbol{a}\cdot\boldsymbol{a}) - 4(\boldsymbol{a}\cdot\boldsymbol{a}) + 2(\boldsymbol{a}\cdot\boldsymbol{b}) - 5(\boldsymbol{a}\cdot\boldsymbol{b}) = 0$$

$$(\boldsymbol{a}\cdot\boldsymbol{a}) = -3(\boldsymbol{a}\cdot\boldsymbol{b})$$

Note that the dot product is negative. This will be important as it allows us to take square roots of negative numbers multiplied by the dot product. Then we substitute into form for the norm of b

$$\boldsymbol{b}\cdot\boldsymbol{b} = -6(\boldsymbol{a}\cdot\boldsymbol{b}) - \boldsymbol{a}\cdot\boldsymbol{b} = -7(\boldsymbol{a}\cdot\boldsymbol{b})$$

Then we write

$$\boldsymbol{a}\cdot\boldsymbol{b} = \sqrt{3(-\boldsymbol{a}\cdot\boldsymbol{b})}\sqrt{7(-\boldsymbol{a}\cdot\boldsymbol{b})}\cos\theta$$

or equivalently

$$1 = \sqrt{21}\cos\theta$$

so $\cos\theta = 1/\sqrt{21}$, giving $\theta = 77.4°$

Exercise 9F

1 a $\boldsymbol{d} = (2-0)\boldsymbol{i} + (3-0)\boldsymbol{j} = 2\boldsymbol{i} + 3\boldsymbol{j}$

Then the vector equation of the line is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + k\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

b $\boldsymbol{d} = (-1-2)\boldsymbol{i} + (3-1)\boldsymbol{j} = -3\boldsymbol{i} + 2\boldsymbol{j}$

Then the vector equation of the line is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} + k\begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

c $\boldsymbol{d} = (3+2)\boldsymbol{i} + (-6+5)\boldsymbol{j} = 5\boldsymbol{i} - \boldsymbol{j}$

Then the vector equation of the line is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} + k\begin{pmatrix} 5 \\ -1 \end{pmatrix}$$

d $\boldsymbol{d} = \left(-\frac{1}{2} - \frac{2}{3}\right)\boldsymbol{i} + \left(\frac{3}{4} + 1\right)\boldsymbol{j} = -\frac{7}{6}\boldsymbol{i} + \frac{7}{4}\boldsymbol{j}$

Then the vector equation of the line is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{3}{4} \end{pmatrix} + k\begin{pmatrix} -\frac{7}{6} \\ \frac{7}{4} \end{pmatrix}$$

2 **a** $\boldsymbol{p} = \boldsymbol{a} - \lambda\boldsymbol{d} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -7 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\Rightarrow \begin{cases} x = 2 + \lambda \\ y = -7 + \lambda \end{cases} \Rightarrow \begin{cases} x - 2 = \lambda \\ y + 7 = \lambda \end{cases}$$

$$\Rightarrow x - 2 = y + 7 \Rightarrow y = x - 9$$

b $\boldsymbol{n} \cdot (\boldsymbol{p} \cdot \boldsymbol{a}) = 0 \Rightarrow \boldsymbol{n} \cdot \boldsymbol{p} = \boldsymbol{n} \cdot \boldsymbol{a}$

$$\Rightarrow \begin{pmatrix} 2 \\ -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}\begin{pmatrix} 2 \\ -7 \end{pmatrix}$$

$$\Rightarrow 2x - 3y = 4 + 21$$

$$2x - 3y = 25$$

$$y = \frac{2x - 25}{3}$$

3 **a** We obtain the direction vector of L,

$$\frac{x-3}{2} = \frac{y+1}{-3}$$

Then

$$-3x + 9 = 2y + 2 = \lambda$$

Then

$$-3x+9=\lambda \Rightarrow x=3-\frac{1}{3}\lambda$$

and

$$2y+2=\lambda \Rightarrow y=-1+\frac{1}{2}\lambda$$

Hence the direction vector is $\boldsymbol{d}=-\frac{1}{3}\boldsymbol{i}+\frac{1}{2}\boldsymbol{j}$

Then the vector equation parallel to L and passing through T is

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -3 \\ 8 \end{pmatrix}+\lambda\begin{pmatrix} -\frac{1}{3} \\ \frac{1}{2} \end{pmatrix}$$

b The perpendicular line must have a normal vector for its direction vector

$$\boldsymbol{d}=-\frac{1}{3}\boldsymbol{i}+\frac{1}{2}\boldsymbol{j} \Rightarrow \boldsymbol{n}=\frac{1}{2}\boldsymbol{i}+\frac{1}{3}\boldsymbol{j}$$

Then the vector equation perpendicular to L passing through T is

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -3 \\ 8 \end{pmatrix}+\lambda\begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \end{pmatrix}$$

c We find the intersection between L and $\boldsymbol{r}$ as

$$\boldsymbol{r}=\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 \\ -1 \end{pmatrix}+\lambda\begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

We write this in Cartesian notation

$$x=1+2\lambda$$

$$y=-1+4\lambda$$

Then

$$\frac{x-1}{2}=\frac{y+1}{4} \Rightarrow 4x-4=2y+2 \Rightarrow y=2x-3$$

We write L in Cartesian form and get

$$\frac{x-3}{2}=\frac{y+1}{-3} \Rightarrow -3x+9=2y+2 \Rightarrow y=\frac{7}{2}-\frac{3}{2}x$$

To find the intersection between the two, we equate both lines and get

$$2x-3=\frac{7}{2}-\frac{3}{2}x \Rightarrow x=\frac{13}{7}$$

and

$$y=2\left(\frac{13}{7}\right)-3=\frac{5}{7}$$

So we must find a line passing through T and $\left(\frac{13}{7}, \frac{5}{7}\right)$, so be obtain the direction vector as

$$\boldsymbol{d} = \left(-3 - \frac{13}{7}\right)\boldsymbol{i} + \left(8 - \frac{5}{7}\right)\boldsymbol{j} = -\frac{34}{7}\boldsymbol{i} + \frac{51}{7}\boldsymbol{j}$$

Then the equation of the line passing through T and the intersection between the two lines is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} -\frac{34}{7} \\ \frac{51}{7} \end{pmatrix}$$

4 a For them to be parallel, their direction vectors have to be proportional to each other.

Note that

$$\boldsymbol{r} = 3\boldsymbol{i} + 2\boldsymbol{j} + \lambda(-\boldsymbol{i} + a\boldsymbol{j}) \Rightarrow \boldsymbol{d}_1 = -\boldsymbol{i} + a\boldsymbol{j}$$

$$\boldsymbol{p} = (1 + 2\mu)\boldsymbol{i} + (5\mu - 2)\boldsymbol{j} \Rightarrow \boldsymbol{d}_2 = 2\boldsymbol{i} + 5\boldsymbol{j}$$

For them to be parallel, we must have

$$\boldsymbol{d}_1 = \gamma \boldsymbol{d}_2$$

for real γ. Then the normalised

$$-\boldsymbol{i} + a\boldsymbol{j} = \gamma(2\boldsymbol{i} + 5\boldsymbol{j})$$

Note that $2\gamma = -1$ gives $\gamma = \frac{-1}{2}$, so

$$a = 5 \times \frac{-1}{2} = -\frac{5}{2}$$

b For them to be perpendicular, their scalar product must be zero, so

$$\boldsymbol{d}_1 \cdot \boldsymbol{d}_2 = 0 \Rightarrow -1 \times 2 + a \times 5 = 0 \Rightarrow 5a = 2 \Rightarrow a = \frac{2}{5}$$

Exercise 9G

1 a $\boldsymbol{d} = (4-1)\boldsymbol{i} + (2-3)\boldsymbol{j} + (1+2)\boldsymbol{k} = 3\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$

Then we write the vector equation simply as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}$$

b $\boldsymbol{d} = (5-3)\boldsymbol{i} + (7-0)\boldsymbol{j} + (-2+5)\boldsymbol{k} = 2\boldsymbol{i} + 7\boldsymbol{j} + 3\boldsymbol{k}$

Then we write the vector equation simply as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 7 \\ 3 \end{pmatrix}$$

2 **a** We write the form for $\boldsymbol{r}$ as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

We substitute with P, and get

$0 = 3 - \lambda$

$2 = -1 + \lambda$

$5 = 2 + 3\lambda$

From the first equations, $\lambda = 3$, and then substituting with that value of λ in the last one gives

$2 + 3(3) = 11 \neq 5$

Hence there is a contradiction, and so P does not lie on the line.

b A parallel line has the same direction vector, and now the equation of the line is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

c We substitute T into the equation of the line to get the system of equations

$-2 = 3 - \lambda$

$4 = -1 + \lambda$

$a = 2 + 3\lambda$

Then $\lambda = 5$, and is consistent in the first two equations. Then

$a = 2 + 3(5) = 17$

3 **a** If the lines are parallel, their direction vectors are proportional to each other. We obtain them by rewriting in the equations for the lines in vector form

$$L_1 : \frac{x-1}{2} = \frac{y-2}{3} = \frac{z+3}{5} = \lambda$$

so

$x = 2\lambda + 1$

$y = 3\lambda + 2$

$z = 5\lambda - 3$

so

$\boldsymbol{d}_1 = 2\boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k}$

and

$$L_2 : x + 2 = \frac{y-1}{-2} = \frac{z-2}{4} = \mu$$

so

$x = \mu - 2$

$y = -2\mu + 1$

$z = 4\mu + 2$

so

$\boldsymbol{d}_2 = \boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}$

The lines are not parallel as there is no real γ for which

$\boldsymbol{d}_1 = \gamma \boldsymbol{d}_2$

b The lines are skew if they are not parallel or perpendicular to each other. We check the scalar product between their direction vectors:

$d_1 \cdot d_2 = 2 \cdot 1 + 3 \cdot (-2) + 5 \cdot 4 = 16 \neq 0$

Hence the lines are skew.

4 a We rewrite the lines in parametric form

$$L_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$$

$$L_2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$$

At the intersection, both lines will take on the same values, so we construct the system of equations

$3 + 5\lambda = 7 - \mu$

$-2 + 4\lambda = 4 + 2\mu$

$1 + 3\lambda = -1 - 3\mu$

From the first equation we get that

$\mu = -3 - 5\lambda + 7 = -5\lambda + 4$

and so

$-2 + 4\lambda = 4 + 2(-5\lambda + 4) \Rightarrow 14\lambda = 14 \Rightarrow \lambda = 1$

so

$\mu = -5(1) + 4 = -1$

We check that these values satisfy the third equation

$1 + 3(1) \neq -1 - 3(-1)$

so the lines do not intersect.

b We rewrite the lines in parametric form

$$L_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$L_2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -7 \\ 0 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$$

At the intersection, both lines will take on the same values, so we construct the system of equations

$\lambda = -7 + 3\mu$

$-1 + 2\lambda = \mu$

$3 - \lambda = 7 - 2\mu$

We substitute the first equation into the second equation and get

$-1 + 2(-7 + 3\mu) = \mu$

$-1 - 14 + 6\mu = \mu$

$5\mu = 15$

$\mu = 3$

and so

$\lambda = -7 + 3(3) = 2$

which is consistent with the third equation.

c We rewrite the lines in parametric form

$$L_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$$

$$L_2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

At the intersection, both lines will take on the same values, so we construct the system of equations

$2\lambda = 1 + 3\mu$

$2 + 5\lambda = -1 + 2\mu$

$4\lambda = -3 + \mu$

The first equation gives us that

$$\lambda = \frac{1 + 3\mu}{2}$$

We substitute this into the third equation and get

$$4\left(\frac{1+3\mu}{2}\right)+3=\mu$$

$$2+6\mu+3=\mu$$

$$5\mu=-5$$

$$\mu=-1$$

and so

$$\lambda=\frac{1+3(-1)}{2}=-1$$

which is consistent with the second equation.

5 First we find the point of intersection between L_1 and L_2

$$L_1:\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-5\\-3\\5\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-1\end{pmatrix}$$

$$L_2:\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-3\\-1\\3\end{pmatrix}+\mu\begin{pmatrix}4\\1\\-2\end{pmatrix}$$

At the intersection, both lines will take on the same values, so we construct the system of equations

$$5+2\lambda=-3+4\mu$$

$$-3+\lambda=-1+\mu$$

$$5-\lambda=3-2\mu$$

The first equation gives us that

$$\mu=\frac{4+\lambda}{2}$$

We substitute this into the third equation and get

$$5-\lambda=3-2\left(\frac{4+\lambda}{2}\right)$$

$$\Rightarrow 5-\lambda=-(1+\lambda)$$

This is not possible, thus the lines are not concurrent.

Exercise 9H

1 a We use the provided formula

$$\begin{pmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{pmatrix}=\begin{pmatrix}3\times3-(-5)\times(-2)\\(-5)\times1-2\times3\\2\times(-2)-3\times1\end{pmatrix}=\begin{pmatrix}9-10\\-5-6\\-4-3\end{pmatrix}=\begin{pmatrix}-1\\-11\\-7\end{pmatrix}$$

b $\begin{pmatrix} 1\times(-2)-0\times 0 \\ 0\times 3-1\times(-2) \\ 1\times 0-1\times 3 \end{pmatrix} = \begin{pmatrix} -2-0 \\ 0+2 \\ 0-3 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix}$

c $\begin{pmatrix} -4\times 2-(-1)\times 1 \\ -1\times 2-3\times 2 \\ 3\times 1-(-4)\times 2 \end{pmatrix} = \begin{pmatrix} -8+1 \\ -2-6 \\ 3+8 \end{pmatrix} = \begin{pmatrix} -7 \\ -8 \\ 11 \end{pmatrix}$

d $\begin{pmatrix} -\frac{3}{4}\times(-2)-1\times\left(-\frac{2}{3}\right) \\ 1\times 1-\left(\frac{1}{2}\right)\times(-2) \\ \left(\frac{1}{2}\right)\times\left(-\frac{2}{3}\right)-\left(-\frac{3}{4}\right)\times 1 \end{pmatrix} = \begin{pmatrix} \frac{6}{4}+\frac{2}{3} \\ 1+1 \\ -\frac{2}{6}+\frac{3}{4} \end{pmatrix} = \begin{pmatrix} \frac{13}{6} \\ 2 \\ \frac{5}{12} \end{pmatrix}$

2 $A = |\boldsymbol{a}\times\boldsymbol{b}|$

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{pmatrix} 3\times(-4)-(-6)\times(-1) \\ (-6)\times 3-2\times(-4) \\ 2\times(-1)-3\times 3 \end{pmatrix} = \begin{pmatrix} -12-6 \\ -18+8 \\ -2-9 \end{pmatrix} = \begin{pmatrix} -18 \\ -10 \\ -11 \end{pmatrix}$$

Then

$$|\boldsymbol{a}\times\boldsymbol{b}| = \sqrt{18^2+10^2+11^2} = 23.3$$

3 **a** $\boldsymbol{AB} = (-2-1)\boldsymbol{i}+(0-4)\boldsymbol{j}+(3-2)\boldsymbol{k} = -3\boldsymbol{i}-4\boldsymbol{j}+\boldsymbol{k}$

$\boldsymbol{AC} = (-1-1)\boldsymbol{i}+(2-4)\boldsymbol{j}+(4-2)\boldsymbol{k} = -2\boldsymbol{i}-2\boldsymbol{j}+2\boldsymbol{k}$

b $A = \frac{1}{2}|\boldsymbol{AB}\times\boldsymbol{AC}|$

$$\boldsymbol{AB}\times\boldsymbol{AC} = \begin{pmatrix} -4\times 2-1\times -2 \\ 1\times(-2)-(-3)\times 2 \\ -3\times(-2)-(-4)(-2) \end{pmatrix} = \begin{pmatrix} -8+2 \\ -2+6 \\ 6-8 \end{pmatrix} = \begin{pmatrix} -6 \\ 4 \\ -2 \end{pmatrix}$$

Then

$$|\boldsymbol{AB}\times\boldsymbol{AC}| = \sqrt{6^2+4^2+2^2} = \sqrt{36+16+4} = 7.48$$

Then the area is

$$A = \frac{1}{2}(7.48) = 3.74$$

4 **i** Let the vectors $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$ be well defined. Then

$$\boldsymbol{a}\times\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta = -|\boldsymbol{a}||\boldsymbol{b}|\sin(-\theta) = -|\boldsymbol{b}||\boldsymbol{a}|\sin(-\theta) = -(\boldsymbol{b}\times\boldsymbol{a})$$

where we have used the commutativity of real numbers and properties of sines. Note that if the angle from a to b is θ, then the angle from b to a is $-\theta$

ii We calculate

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=\begin{pmatrix} a_2(b_1c_2-b_2c_1)+a_3(b_1c_3+a_3c_1) \\ a_3(a_2b_3-a_3b_2)-a_1(b_1c_2-b_2c_1) \\ -a_1(b_1c_3+b_3c_1)-a_2(a_2b_3-a_3b_2) \end{pmatrix}=(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$$

iii $$\lambda\begin{pmatrix} a_2b_3-a_3b_2 \\ a_3b_1-a_1b_3 \\ a_1b_2-a_2b_1 \end{pmatrix}=\begin{pmatrix} \lambda(a_2b_3-a_3b_2) \\ \lambda(a_3b_1-a_1b_3) \\ \lambda(a_1b_2-a_2b_1) \end{pmatrix}=\begin{pmatrix} a_2(\lambda b_3)-a_3(\lambda b_2) \\ a_3(\lambda b_1)-a_1(\lambda b_3) \\ a_1(\lambda b_2)-a_2(\lambda b_1) \end{pmatrix}=\begin{pmatrix} (\lambda a_2)b_3-(\lambda a_3)b_2 \\ (\lambda a_3)b_1-(\lambda a_1)b_3 \\ (\lambda a_1)b_2-(\lambda a_2)b_1 \end{pmatrix}$$

Hence

$\lambda(a\times b)=(\lambda a)\times b=a\times(\lambda b)$, $\lambda\in\mathrm{R}$

iv We can expand out the cross products explicitly as

$$(\boldsymbol{a}+\boldsymbol{b})\times\boldsymbol{c}=\begin{pmatrix} (a_2+b_2)c_3-(a_3+b_3)c_2 \\ (a_3+b_3)c_1-(a_1+b_1)c_3 \\ (a_1+b_1)c_2-(a_2+b_2)c_1 \end{pmatrix}=\begin{pmatrix} a_2c_3-a_3c_2 \\ a_3c_1-a_1c_3 \\ a_1c_2-a_2c_1 \end{pmatrix}+\begin{pmatrix} b_2c_3-b_3c_2 \\ b_3c_1-b_1c_3 \\ b_1c_2-b_2c_1 \end{pmatrix}=(\boldsymbol{a}\times\boldsymbol{c})+(\boldsymbol{b}\times\boldsymbol{c})$$

5 We write out the vectors

$$\boldsymbol{AB}=(2-1)\boldsymbol{i}+(-1-1)\boldsymbol{j}+(0-1)\boldsymbol{k}=\boldsymbol{i}-2\boldsymbol{j}-\boldsymbol{k}$$

$$\boldsymbol{AC}=(2-1)\boldsymbol{i}+(4-1)\boldsymbol{j}+(2-1)\boldsymbol{k}=\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k}$$

$$\boldsymbol{AD}=(-2-1)\boldsymbol{i}+(2-1)\boldsymbol{j}+(2-1)\boldsymbol{k}=-3\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$$

Then

$$(\boldsymbol{AB}\times\boldsymbol{AC})\cdot\boldsymbol{AD}=\begin{pmatrix} (-2)\times1-(-1)\times3 \\ -1\times1-1\times1 \\ 1\times3-(-2)\times1 \end{pmatrix}\cdot\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}=\begin{pmatrix} -2+3 \\ -1-1 \\ 3+2 \end{pmatrix}\cdot\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}=\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}\cdot\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

$$=1\times(-3)+(-2)\times(1)+5\times(1)=0$$

Hence the three points are coplanar.

6 **a** We find D such that AB = DC. Then

$$AB=\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}-\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}=\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$$

and

$$DC=\begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix}-\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}=\begin{pmatrix} 4-d_1 \\ 5-d_2 \\ -1-d_3 \end{pmatrix}$$

Then we have the equations

$$4-d_1=1$$

$$5-d_2=-3$$

$$-1-d_3=2$$

Then

$D=(3,8,-3)$

b Note that the vectors DC, DA and DH enclose the parallelepiped, so

$$\boldsymbol{DC}=\begin{pmatrix}1\\-3\\2\end{pmatrix}\cdot\begin{pmatrix}i\\j\\k\end{pmatrix}$$

$$\boldsymbol{DA}=\begin{pmatrix}1-3\\2-8\\1+3\end{pmatrix}\cdot\begin{pmatrix}i\\j\\k\end{pmatrix}=\begin{pmatrix}-2\\-6\\4\end{pmatrix}\cdot\begin{pmatrix}i\\j\\k\end{pmatrix}$$

$$\boldsymbol{DH}=\begin{pmatrix}4-3\\3-8\\6+3\end{pmatrix}\cdot\begin{pmatrix}i\\j\\k\end{pmatrix}=\begin{pmatrix}1\\-5\\9\end{pmatrix}\cdot\begin{pmatrix}i\\j\\k\end{pmatrix}$$

c The volume of the parallelepiped is given by

$$V=\left|(\boldsymbol{DC}\times\boldsymbol{DA})\cdot\boldsymbol{DH}\right|=\left|(0)(1)+(-8)(-5)+(-12)(9)\right|=68$$

7 Assuming D is the apex, we obtain

$$\boldsymbol{BC}=\boldsymbol{BD}+\boldsymbol{DC}=(1-2)\boldsymbol{i}+(-2+1)\boldsymbol{j}+(4+5)\boldsymbol{k}=-\boldsymbol{i}-\boldsymbol{j}+9\boldsymbol{k}$$

$$V=\frac{1}{3}Base\cdot h=\frac{1}{3}\cdot\frac{1}{2}\left|\boldsymbol{BA}\times\boldsymbol{BD}\right|\left|\boldsymbol{BC}\right|$$

$$\boldsymbol{BA}\times\boldsymbol{BD}=\begin{pmatrix}-3\times4-2\times(-2)\\2\times1-(-2)\times4\\-2\times(-2)-(-3)\times1\end{pmatrix}=\begin{pmatrix}-12+4\\2+8\\4+3\end{pmatrix}=\begin{pmatrix}-8\\10\\7\end{pmatrix}$$

Then

$$V=\frac{1}{6}\left(\sqrt{8^2+10^2+7^2}\sqrt{1^2+1^2+9^2}\right)=22.2$$

8 $(\boldsymbol{a}\cdot\boldsymbol{b})^2+\left|(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{a}\times\boldsymbol{b})\right|=|\boldsymbol{a}|^2|\boldsymbol{b}|^2\cos^2\theta+|\boldsymbol{a}|^2|\boldsymbol{b}|^2\sin^2\theta=|\boldsymbol{a}|^2|\boldsymbol{b}|^2(\cos^2\theta+\sin^2\theta)=|\boldsymbol{a}|^2|\boldsymbol{b}|^2=(|\boldsymbol{a}||\boldsymbol{b}|)^2$

9 We calculate

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=\begin{pmatrix}a_2(b_1c_2-b_2c_1)+a_3(b_1c_3+a_3c_1)\\a_3(a_2b_3-a_3b_2)-a_1(b_1c_2-b_2c_1)\\-a_1(b_1c_3+b_3c_1)-a_2(a_2b_3-a_3b_2)\end{pmatrix}=(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{b})$$

Exercise 9I

1 a $\boldsymbol{p}=\boldsymbol{a}+\lambda\boldsymbol{u}+\mu\boldsymbol{v}=\begin{pmatrix}0\\2\\-1\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\4\end{pmatrix}+\mu\begin{pmatrix}-1\\2\\-1\end{pmatrix}$

b $\boldsymbol{p} = \boldsymbol{a} + \lambda\boldsymbol{u} + \mu\boldsymbol{v} = \begin{pmatrix}1\\-2\\3\end{pmatrix} + \lambda\begin{pmatrix}0\\-3\\2\end{pmatrix} + \mu\begin{pmatrix}1\\4\\-2\end{pmatrix}$

c $\boldsymbol{p} = \boldsymbol{a} + \lambda\boldsymbol{u} + \mu\boldsymbol{v} = \begin{pmatrix}-3\\4\\2\end{pmatrix} + \lambda\begin{pmatrix}-2\\0\\3\end{pmatrix} + \mu\begin{pmatrix}2\\1\\5\end{pmatrix}$

2 a $\boldsymbol{AB} = (-1-0)\boldsymbol{i} + (2-1)\boldsymbol{j} + (0-3)\boldsymbol{k} = -\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$

$\boldsymbol{AC} = (3-0)\boldsymbol{i} + (-2-1)\boldsymbol{j} + (4-3)\boldsymbol{k} = 3\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$

Then we can write the vector equation of the plane as

$$p = \begin{pmatrix}0\\1\\3\end{pmatrix} + \lambda\begin{pmatrix}-1\\1\\-3\end{pmatrix} + \mu\begin{pmatrix}3\\-3\\1\end{pmatrix}$$

b $x = -\lambda + 3\mu$

$y = 1 + \lambda - 3\mu$

$z = 3 - 3\lambda + \mu$

c We eliminate the parameters in **b**

$\lambda = 3\mu - x$

which we substitute into the equation for y and z to get

$y = 1 + 3\mu - x - 3\mu \Rightarrow y + x = 1$

$z = 3 - 3(3\mu - x) + \mu$

We cannot express the equation in terms of x, y and z, so the Cartesian equation is

$y + x = 1$

3 a The normal vector is

$$\begin{pmatrix}2\\1\\4\end{pmatrix} \times \begin{pmatrix}-1\\2\\-1\end{pmatrix} = \begin{pmatrix}-9\\-2\\5\end{pmatrix}$$

$$\begin{pmatrix}-9\\-2\\5\end{pmatrix} \cdot \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-9\\-2\\5\end{pmatrix} \cdot \begin{pmatrix}0\\2\\-1\end{pmatrix}$$

$-9x - 2y + 5z = -4 - 5$

$-9x - 2y + 5z = -9$

b $\begin{pmatrix}0\\-3\\2\end{pmatrix} \times \begin{pmatrix}1\\4\\-2\end{pmatrix} = \begin{pmatrix}-2\\2\\3\end{pmatrix}$

$$\begin{pmatrix}-2\\2\\3\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-2\\2\\3\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\3\end{pmatrix}$$

$$-2x+2y+3z=-2-4+9$$

$$-2x+2y+3z=3$$

c $$\begin{pmatrix}-2\\0\\3\end{pmatrix}\times\begin{pmatrix}2\\1\\5\end{pmatrix}=\begin{pmatrix}-3\\16\\-2\end{pmatrix}$$

$$\begin{pmatrix}-3\\16\\-2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-3\\16\\-2\end{pmatrix}\cdot\begin{pmatrix}-3\\4\\2\end{pmatrix}$$

$$-3x+16y-2z=69$$

4 **a** We substitute the point into the equation of the plane

$$3(5)-4(4)+2(-2)=-5\neq 5$$

Hence the point is not on the plane

b The normal vector is

$$\boldsymbol{d}=3\boldsymbol{i}-4\boldsymbol{j}+2\boldsymbol{k}$$

Then we are searching for a plane with the same normal vector but a different point.

$$\begin{pmatrix}3\\-4\\2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3\\-4\\2\end{pmatrix}\cdot\begin{pmatrix}5\\4\\-2\end{pmatrix}$$

$$3x-4y+2z=15-16-4$$

$$3x-4y+2z=-5$$

5 We equate solve the equations of a plane as a system of equations

$$x+y-z=1$$

$$2x-3y-9z=10$$

$$x+2y-3z=-4$$

We subtract the third from the first and get

$$-y+2z=5$$

$$y=2z-5$$

We subtract two times the first from the second, and get

$$-5y-7z=8$$

Then substituting our value for y we get that

$$-5(2z-5)-7z=8$$

$z = 1$

Then

$y = 2(1) - 5 = -3$

and so

$x + (-3) - 1 = 1$

$x = 5$

6 **a** We express y and z in terms of x

$y = 3 + 2z - x$

$z = 1 + 3y - 2x$

Then

$y = 3 + 2(1 + 3y - 2x) - x$
$y = x - 1$

Which we can then substitute into the equation for z as

$z = 1 + 3(x - 1) - 2x$

this simplifies into

$z = x - 2$

We let $x = \lambda$, and so

$x = \lambda$

$y = \lambda - 1$

$z = \lambda - 2$

We eliminate λ to find the Cartesian equation, as

$x = y + 1 = z + 2$

b We can set the new equation to be generated by

$A = \lambda \boldsymbol{i} + (\lambda - 1)\boldsymbol{j} + (\lambda - 2)\boldsymbol{k}$

and

$T = 2\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$

so we can write it as

$$p = \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} \Rightarrow 5x + y - 6z = -13$$

7 If two planes are parallel, their normal vectors are parallel, then

$n_1 \times n_2 = |n_1||n_2|\sin\theta = |n_1||n_2| \times 0 = 0$

If the vector product of the normal vectors is zero, we have

$n_1 \times n_2 = |n_1||n_2|\sin\theta = 0 \Rightarrow \theta = 0$

hence they are parallel

Exercise 9J

1 a $x - 5 = \lambda$

$\frac{y+1}{2} = \lambda$

$\frac{1-z}{3} = \lambda$

or equivalently

$x = \lambda + 5$

$y = 2\lambda - 1$

$z = 1 - 3\lambda$

We substitute in the equation of the plane

$2(\lambda + 5) - 4(2\lambda - 1) + 1 - 3\lambda = -3$

$2\lambda + 10 - 8\lambda + 4 + 1 - 3\lambda = -3$

$-9\lambda = -18$

$\lambda = 2$

There is a unique solution, so the line and the plane intersect at a point. This point is

$x = 2 + 5 = 7$

$y = 2(2) - 1 = 3$

$z = 1 - 3(2) = -5$

So they intersect at $(7, 3, -5)$.

b $1 - 2x = \lambda$

$\frac{y-3}{4} = \lambda$

$\frac{2z+2}{3} = \lambda$

or equivalently

$x = \frac{1-\lambda}{2}$

$y = 4\lambda + 3$

$z = \frac{3\lambda - 2}{2}$

We substitute in the equation of the plane

$5\left(\frac{1-\lambda}{2}\right) + (4\lambda + 3) - 4\left(\frac{3\lambda - 2}{2}\right) = 3$

$\lambda = \frac{13}{9}$

There is a unique solution, so the line and the plane intersect at a point. This point is determined by

$x = -\frac{2}{9}$

$y = \frac{79}{9}$

$z = \frac{7}{6}$

c $\frac{x-5}{4} = \lambda$

$\frac{y+2}{-2} = \lambda$

$\frac{z-3}{3} = \lambda$

or equivalently

$x = 4\lambda + 5$

$y = -2\lambda - 2$

$z = 3\lambda + 3$

We substitute in the equation of the plane

$2(4\lambda + 5) + (-2\lambda - 2) - 2(3\lambda + 3) = 3$

This has no solutions, so there is no intersection.

d $\frac{1-x}{2} = \lambda$

$\frac{y+2}{3} = \lambda$

$1 - 3z = \lambda$

or equivalently

$x = 1 - 2\lambda$

$y = 3\lambda - 2$

$z = \frac{1-\lambda}{3}$

We substitute in the equation of the plane

$$2(1-2\lambda)+(3\lambda-2)-3\left(\frac{1-\lambda}{3}\right)=-1$$

This has infinite solutions, so the line is contained in the plane.

2 The normal of the plane and the direction vector of the line must be orthogonal, so their dot product must be zero. We obtain the parametric equation of the line as

$$\frac{x}{m}=\lambda \Rightarrow x=\lambda m$$

$$\frac{y-1}{2}=\lambda \Rightarrow y=2\lambda+1$$

$$\frac{z+2}{4}=\lambda \Rightarrow z=4\lambda-2$$

Then

$$\boldsymbol{d}=\begin{pmatrix} m \\ 2 \\ 4 \end{pmatrix}$$

and the normal of the plane is

$$\boldsymbol{n}=\begin{pmatrix} 2 \\ m \\ -3 \end{pmatrix}$$

Then

$$\boldsymbol{d}\cdot\boldsymbol{n}=\begin{pmatrix} m \\ 2 \\ 4 \end{pmatrix}\cdot\begin{pmatrix} 2 \\ m \\ -3 \end{pmatrix}=2m+2m-12=0$$

gives

$$m=3$$

3 This is precisely what we have calculated above, as

$$\boldsymbol{d}\cdot\boldsymbol{n}=|\boldsymbol{d}||\boldsymbol{n}|\cos\theta$$

and $\theta=\pi/2$, so $\boldsymbol{d}\cdot\boldsymbol{n}=0$.

Exercise 9K

1 a Let $x=\lambda$

$$3\lambda+y-2z=-1$$

$$\lambda-4y+2z=3$$

Then

$$y=\frac{4\lambda-2}{3}$$

and

$$z = \frac{1+13\lambda}{6}$$

which determine the equation of the line

b $n_1 = (3,1,-2)$

$n_2 = (1,-4,2)$

$$\cos\theta = \frac{|n_1 \cdot n_2|}{|n_1||n_2|} = \frac{|3(1)+(1)(-4)+(-2)(2)|}{\sqrt{3^2+1^2+2^2}\sqrt{1^2+4^2+2^2}} = \frac{5}{7\sqrt{6}}$$

Then

$$\theta = \cos^{-1}\frac{5}{7\sqrt{6}} = 1.275$$

2 Any system of equations formed has no solution, so these lines do not intersect.

3 **a** We have that

$x = \lambda + 2$

$y = -3\lambda + 1$

$z = 2\lambda + 2$

Substitute in the equation of the plane as

$3(\lambda+2)+2(-3\lambda+1)-(2\lambda+2)=1$

$\lambda = 1$

Then the point of intersection is $P = (3,-2,4)$

b The direction vector of the line is

$d = i - 3j + 2k$

The normal vector of the plane is

$n = 3i + 2j - k$

Then

$$\sin\theta = \frac{|d \cdot n|}{|d||n|} = \frac{|(1)(3)+(-3)(2)+(2)(-1)|}{\sqrt{1^2+3^2+2^2}\sqrt{3^2+2^2+1^2}} = \frac{5}{14}$$

$\theta = 0.365$

4 We look at the angle between the normal vectors

$n_1 = (a,0,a)$

and

$n_2 = (b,-b,0)$

Note that

$n_1 \cdot n_2 = a \cdot b$

$|n_1| = \sqrt{2}|a|$

$|n_2| = \sqrt{2}|b|$

$ab = 2|a||b|\cos\theta$

so

$\cos\theta = \pm\frac{1}{2}$

It is the angle between their normal vectors if it is acute and it is the supplementary angle if it is obtuse, hence for both the positive and the negative case, the angle will be $\frac{\pi}{3}$

Exercise 9L

1 a For A, we have that the direction vector will be

$$\boldsymbol{d}_a = (0+3000)\boldsymbol{i} + (0-5000)\boldsymbol{j}$$

and since the speed is 4m/s we have to normalise and multiply by this so that the magnitude holds. Then

$$\boldsymbol{d}_a = \frac{4\times 3000\boldsymbol{i} - 4\times 5000\boldsymbol{j}}{\sqrt{3000^2+5000^2}} = \frac{12}{\sqrt{34}}\boldsymbol{i} - \frac{20}{\sqrt{34}}\boldsymbol{j}$$

and so with the point $(-3000, 5000)$, the equation of the position becomes

$$\boldsymbol{a} = (-3000\boldsymbol{i} + 5000\boldsymbol{j}) + t\left(\frac{12}{\sqrt{34}}\boldsymbol{i} - \frac{20}{\sqrt{34}}\boldsymbol{j}\right)$$

similarly for L we have

$$\boldsymbol{d}_l = (0-7000)\boldsymbol{i} + (0-9000)\boldsymbol{j}$$

and since the speed is 4 ms^{-1} we have to normalise and multiply by this so that the magnitude holds. Then

$$\boldsymbol{d}_a = \frac{6\times(-7000)\boldsymbol{i} - 6\times(-9000)\boldsymbol{j}}{\sqrt{7000^2+9000^2}} = -\frac{42}{\sqrt{130}}\boldsymbol{i} - \frac{54}{\sqrt{130}}\boldsymbol{j}$$

and so with the point $(7000, 9000)$, the equation of the position becomes

$$\boldsymbol{l} = (7000\boldsymbol{i} + 9000\boldsymbol{j}) + t\left(-\frac{42}{\sqrt{130}}\boldsymbol{i} - \frac{54}{\sqrt{130}}\boldsymbol{j}\right)$$

b We check when each boat gets to the point (0,0). For A

$$x = -3000 + \frac{12}{\sqrt{34}}t$$

$$y = 5000 - \frac{20}{\sqrt{34}}t$$

Then at (0,0)

$$-3000+\frac{12}{\sqrt{34}}t=5000-\frac{20}{\sqrt{34}}t$$

$$\frac{32}{\sqrt{34}}t=2000$$

$$t=\frac{125\sqrt{34}}{2}\approx 364.4s$$

For L we have

$$x=7000-\frac{42}{\sqrt{130}}t$$

$$y=9000-\frac{54}{\sqrt{130}}t$$

Then at (0,0)

$$7000-\frac{42}{\sqrt{130}}t=9000-\frac{54}{\sqrt{130}}t$$

$$\frac{12}{\sqrt{130}}t=2000$$

$$t=500\frac{\sqrt{130}}{3}\approx 1900.3s$$

Boat A will arrive first and the boat L takes more 1535.9 s to reach the boat in need..

2 **a** The initial position is given at time $t=0$ so

$$\boldsymbol{p}(0)=23\boldsymbol{i}+8\boldsymbol{j}+43\boldsymbol{k}$$

b The speed is given by the magnitude of the direction vector

$$|d|=\sqrt{2^2+1^2+4^2}=\sqrt{21}\approx 4.58\,\text{ms}^{-1}$$

c Intersection between the line given and the plane. The components of p are

$$x=23+2t$$

$$y=8-t$$

$$z=43+4t$$

We substitute into the equation of the plane to get

$$12(23+2t)-3(8-t)-5(43+4t)=-2$$

$$276+24t-24+3t-215-20t=-2$$

$$39=-7t$$

$$t=-\frac{39}{7}=-5.57\text{ s}$$

d Total distance = $5.57\times\sqrt{21}\approx 25.5$ m

3 a Assuming distance is in km and time in hours

Speed of p_1

$v_1 = \sqrt{8^2 + 9^2 + 0.25^2} \approx 12.04\ \text{kmh}^{-1}$

Speed of p_2

$v_2 = \sqrt{7^2 + 11^2 + 0.2^2} = 13.04\text{kmh}^{-1}$

b Assume that there is an intersection. We write out the components of p_1 and p_2

$x_1 = 147 - 8t$

$y_1 = -156 + 9t$

$z_1 = 5 + 0.25t$

$x_2 = -118 + 7\mu$

$y_2 = 189 - 11\mu$

$z_2 = 7 + 0.2\mu$

We equate the components to get a value of t

$147 - 8t = -118 + 7\mu$

$-156 + 9t = 189 - 11\mu$

$5 + 0.25t = 7 + 0.2\mu$

This gives $\mu = 15$ and $t = 20$ which is consistent in all three equations. Hence the paths intersect. The point of intersection is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 147 - 8(20) \\ -156 + 9(20) \\ 5 + 0.25(20) \end{pmatrix} = \begin{pmatrix} -13 \\ 24 \\ 10 \end{pmatrix}$$

c The times at which they reach this point are different, and unique. Hence they will not collide.

Chapter review

1 a $\boldsymbol{a} + \boldsymbol{b} = AB$

Hence the midpoint will have half of that length, so

$\boldsymbol{m} = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$

b $AD = \frac{2\sqrt{5}}{3} DC$, so AD and DC are the parallel sides of the trapezium.

c Midpoints are (6, 1), (4.5, 3.5), (2, -1), (8.5, 5.5), which give two pairs of parallel lines with equal length and thus form a rhombus.

2 **a** We calculate the Cartesian form

$x = 2 + \lambda - 3\mu$

$y = 2\lambda + \mu$

$z = \mu - 1$

We subtract the second from twice the first

$2x - y = 4 + 2\lambda - 2\lambda - 6\mu - \mu$

Then

$2x - y = 4 - 7\mu$

We add 7 times the third equation as

$2x - y + 7z = 7\mu + 4 - 7 - 7\mu$

$2x - y + 7z = -3$

b We substitute with each of the points, leading to the equations

$2(2) - (0) + 7(a) = -3$

$2(b) - 4 + 7(-1) = -3$

$2(-1) - d + 7(0) = -3$

Then we solve them and get

$$a = \frac{-3-4}{7} = -1$$

$$b = \frac{-3+4+7}{2} = 4$$

$$d = \frac{-3+2}{-1} = 1$$

c We write (taking all vectors from the origin)

$C = B - A + D = A = B + D - 2A$

$$C = \begin{pmatrix} 4 \\ 4 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 1 \end{pmatrix}$$

d We substitute with the point E and get

$2(1) - (-2) + 7(1) = 11 \neq -3$

so the point E does not lie in the plane

e We use the formula for the volume of the pyramid. We calculate

$$AC = \begin{pmatrix} -1 \\ 5 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 2 \end{pmatrix}$$

$$AB = \begin{pmatrix} 4 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$$

$$AE = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}$$

Then

$$V = \frac{1}{6}\left|(AC \times AB) \cdot AE\right|$$

$$V = \frac{1}{6}\left|(-8)(-1) + (4)(-2) + (-22)(2)\right| = 7.33$$

3 **a** The direction vector of the line will be the normal to the plane

$$d = (2, -2, 1)$$

Then the equation of the line is

$$p = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

b The point of intersection is obtained by substituting

$$x = 2 + 2\lambda$$

$$y = -2 - 2\lambda$$

$$z = 1 + \lambda$$

into the equation of the plane

$$2(2 + 2\lambda) - 2(-2 - 2\lambda) + (1 + \lambda) = 0$$

$$4 + 4\lambda + 4 + 4\lambda + 1 + \lambda = 0$$

$$\lambda = -1$$

Then the point of intersection is

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Then the distance between this point and the plane is

$$|OA| = \sqrt{2^2 + 2^2 + 1^2} = 3$$

c A point on the plane is $B = (0, 0, 0)$ and we define the vector

$$BP = x_0 i + y_0 j + z_0 k$$

The normal of the plane is n

$$n = 2i - 2j + k$$

Then the distance we need is

$$d=\frac{|BP\cdot n|}{|n|}=\frac{|2x_0-2y_0+z_0|}{\sqrt{2^2+2^2+1^2}}=\frac{|2x_0-2y_0+z_0|}{3}$$

4 **a** Note that

$a\cdot b=pr+4+rp$

and

$a\cdot b=\left(p^2+4+r^2\right)\cos\theta$

since the components form an arithmetic sequence with common difference d, we have the relationship

$p+d=2$

$2+d=r$

We use this to rewrite the formula for the dot product in terms of d and get

$$\frac{(2-d)(2-d)+4+(2-d)(2-d)}{(2-d)(2+d)+4+(2-d)(2+d)}=\frac{12-2d^2}{12+2d^2}=\frac{6-d^2}{6+d^2}$$

as required.

b When the angle is 60°, the cosine is $\frac{1}{2}$ so

$$\frac{6-d^2}{6+d^2}=\frac{1}{2}$$

$12-2d^2=6+d^2$

$3d^2=6$

$d^2=2$

Then $d=\pm\sqrt{2}$

5 If these planes are perpendicular, then their normal vectors are always perpendicular, so we check

$n_1\cdot n_2=\sin\alpha\cdot\cos\alpha+\cos\alpha\sin\alpha-1$

These planes are not perpendicular

6 If they are perpendicular, their dot product will be equal to zero. We use the fact that their magnitude is 1 to calculate

$(2\boldsymbol{u}-3\boldsymbol{v})\cdot(5\boldsymbol{u}+2\boldsymbol{v})=10\boldsymbol{u}\cdot\boldsymbol{u}+4\boldsymbol{u}\cdot\boldsymbol{v}-15\boldsymbol{v}\cdot\boldsymbol{u}-6\boldsymbol{v}\cdot\boldsymbol{v}$

$-11\,\boldsymbol{u}\cdot\boldsymbol{v}+4=-11\cos\theta+4=0$

$$\cos\theta=\frac{4}{11}$$

$\theta=69^\circ$

7 **a** $x=3\lambda+4$

$y = 1 - \lambda$

$z = 2\lambda + 5$

$4(3\lambda + 4) - 3(1 - \lambda) + (2\lambda + 5) = 1$

$12\lambda + 16 - 3 + 3\lambda + 2\lambda + 5 = 1$

$\lambda = -1$

Then the point P is at $\lambda = -1$ and so the point is

$x = 3(-1) + 4 = 1$

$y = 1 - (-1) = 2$

$z = 2(-1) + 5 = 3$

b Angle between the line and the plane

$\boldsymbol{d} = (3, -1, 2)$

$\boldsymbol{n} = (4, -3, 1)$

$$\sin\theta = \frac{|\boldsymbol{d} \cdot \boldsymbol{n}|}{|\boldsymbol{d}||\boldsymbol{n}|} = \frac{|(3)(4) + (-1)(-3) + (2)(1)|}{\sqrt{3^2 + 1^2 + 2^2}\sqrt{4^2 + 3^2 + 1^2}} \Rightarrow \theta = 63°$$

8 **a** $\boldsymbol{a} \cdot \boldsymbol{b} = 2^x(2^x) + (4^x)(0.5^x) + (5)(-4) = 0$

$4^x + 2^x - 20 = 0$

This is true for $x = 2$.

b The equation of the plane is given by

$$\boldsymbol{p} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 16 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 0.25 \\ -4 \end{pmatrix}$$

9 **a** We have the following relations

$\boldsymbol{a} \cdot \hat{\boldsymbol{k}} = |\boldsymbol{a}||\hat{\boldsymbol{k}}|\cos\gamma$

$\boldsymbol{a} \cdot \hat{\boldsymbol{j}} = |\boldsymbol{a}||\hat{\boldsymbol{j}}|\cos\beta$

$\boldsymbol{a} \cdot \hat{\boldsymbol{i}} = |\boldsymbol{a}||\hat{\boldsymbol{i}}|\cos\alpha$

Note that the norm of the unit vectors is one, and

$$|\boldsymbol{a}| = \sqrt{(\boldsymbol{a} \cdot \hat{\boldsymbol{k}})^2 + (\boldsymbol{a} \cdot \hat{\boldsymbol{j}})^2 + (\boldsymbol{a} \cdot \hat{\boldsymbol{i}})^2}$$

We substitute with the relations obtained and get

$|\boldsymbol{a}|^2 = |\boldsymbol{a}|^2\cos^2\gamma + |\boldsymbol{a}|^2\cos^2\beta + |\boldsymbol{a}|^2\cos^2\alpha$

$1 = \cos^2\gamma + \cos^2\beta + \cos^2\alpha$

b The norm of a is

$$|\boldsymbol{a}| = \sqrt{3^2 + 6^2 + 2^2} = 7$$

We substitute into the relations obtained in a to get

$$3 = 7\cos\alpha \Rightarrow \alpha = 64.6°$$

$$-6 = 7\cos\beta \Rightarrow \beta = 149°$$

$$2 = 7\cos\gamma \Rightarrow \gamma = 73.4°$$

c When the plane passes through zero, the normal vector will correspond precisely to the unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$. As we saw in **a**, these can be written as the cosines of the angles. Hence

$$\boldsymbol{n} = \cos\alpha\boldsymbol{i} + \cos\beta\boldsymbol{j} + \cos\gamma\boldsymbol{k}$$

Then the equation of the plane can be written as

$$x\cos\alpha + y\cos\beta + z\cos\gamma = 0$$

10 a We calculate the vectors

$$AP = \begin{pmatrix}1\\2\\4\end{pmatrix} - \begin{pmatrix}2\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\2\\4\end{pmatrix}$$

and

$$AQ = \begin{pmatrix}0\\1\\4\end{pmatrix} - \begin{pmatrix}2\\0\\0\end{pmatrix} = \begin{pmatrix}-2\\1\\4\end{pmatrix}$$

These will be the two vectors on the plane equation. Additionally we take a point, choosing for simplicity A = (2,0,0). Then the plane equation in vector form is

$$p = \begin{pmatrix}2\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}-1\\2\\4\end{pmatrix} + \mu\begin{pmatrix}-2\\1\\4\end{pmatrix}$$

To write it in Cartesian form, we write out the system of equations

$$x = 2 - \lambda - 2\mu$$

$$y = 2\lambda + \mu$$

$$z = 4\lambda + 4\mu$$

We subtract the third one from twice the second one, to get

$$z - 2y = 2\mu$$

so

$$\mu = \frac{z - 2y}{2}$$

and we add the second one to twice the first one, to get

$$y + 2x = 4 + 2\lambda - 2\lambda + \mu - 4\mu$$

or equivalently

$y + 2x = 4 - 3\mu$

Then we substitute with our value for μ to get

$y + 2x = 4 - 3\left(\frac{z - 2y}{2}\right)$

This simplifies to

$4x - 4y + 3z = 8$

b using the equation of the plane written in a. BG gives the direction vector of the line.

$$BG = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 4 \end{pmatrix}$$

Then the equation of the line is written as

$$\boldsymbol{p} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} - \lambda \begin{pmatrix} -2 \\ -2 \\ 4 \end{pmatrix}$$

c Angle between plane
$4x - 4y + 3z = 8$

and line

$$\boldsymbol{p} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} - \lambda \begin{pmatrix} -2 \\ -2 \\ 4 \end{pmatrix}$$

We have that

$$\sin\theta = \frac{|\boldsymbol{d} \cdot \boldsymbol{n}|}{|\boldsymbol{d}||\boldsymbol{n}|} = \frac{|(-2)(4) + (-2)(-4) + (4)(3)|}{\sqrt{2^2 + 2^2 + 4^2}\sqrt{4^2 + 4^2 + 3^2}} = \frac{12}{2\sqrt{246}} = \frac{6}{\sqrt{246}}$$

Then

$\theta = 22.5°$

Exam-style questions

11 a $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ (1 mark)

$\overrightarrow{AC} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$ (1 mark)

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$ (1 mark)

$$= \begin{pmatrix} 2 \\ -7 \\ 4 \end{pmatrix}$$ (1 mark)

b $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{2^2 + (-7)^2 + 4^2}$ (2 marks)

$= \frac{\sqrt{69}}{2}$ (1 mark)

c $\boldsymbol{r}.\begin{pmatrix} 2 \\ -7 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}.\begin{pmatrix} 2 \\ -7 \\ 4 \end{pmatrix}$ (2 marks)

$\boldsymbol{r}.\begin{pmatrix} 2 \\ -7 \\ 4 \end{pmatrix} = 6$ (1 mark)

$2x - 7y + 4z = 6$

d $\begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -7 \\ 4 \end{pmatrix} = \begin{pmatrix} -13 \\ -10 \\ -11 \end{pmatrix}$ (2 marks)

$\boldsymbol{n} = \begin{pmatrix} 13 \\ 10 \\ 11 \end{pmatrix}$

$y = 0 \Rightarrow x = -\frac{1}{5}, z = \frac{8}{5}$ (or equivalent) (2 marks)

$\boldsymbol{r} = \begin{pmatrix} -\frac{1}{5} \\ 0 \\ \frac{8}{5} \end{pmatrix} + \lambda\begin{pmatrix} 13 \\ 10 \\ 11 \end{pmatrix}$ (or equivalent) (1 mark)

12 $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}$ (1 mark)

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$ (1 mark)

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ (1 mark)

$\text{Volume} = \frac{1}{6}\left|\overrightarrow{AB}.\overrightarrow{AC} \times \overrightarrow{AD}\right| = \frac{1}{6}\left|\begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}.\begin{pmatrix} 7 \\ 4 \\ -9 \end{pmatrix}\right|$ (2 marks)

$= \frac{1}{6}\left|(-21 + 8 - 18)\right|$

$= \dfrac{31}{6}$ units^2. (1 mark)

13 $\overrightarrow{AP} = \boldsymbol{p} - \boldsymbol{a}$ (1 mark)

$\overrightarrow{BP} = \boldsymbol{p} - \boldsymbol{b}$ (1 mark)

$\overrightarrow{AP}.\overrightarrow{BP} = (\boldsymbol{p} - \boldsymbol{a}).(\boldsymbol{p} - \boldsymbol{b})$ (1 mark)

$= (\boldsymbol{p} - \boldsymbol{a}).(\boldsymbol{p} + \boldsymbol{a})$ (1 mark)

$= \boldsymbol{p.p} - \boldsymbol{a.p} + \boldsymbol{a.p} - \boldsymbol{a.a}$ (1 mark)

$= \boldsymbol{p.p} - \boldsymbol{a.a}$ (1 mark)

$= |\boldsymbol{p}|^2 - |\boldsymbol{a}|^2$ (1 mark)

$= 0$ since $|\boldsymbol{p}| = |\boldsymbol{a}|$ (1 mark)

Therefore $\overrightarrow{AP}$ is perpendicular to $\overrightarrow{BP}$ and $\angle APB = 90^\circ$

14 a Equation of line perpendicular to Π and passing through P is $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$

(2 marks)

Attempting to solve P and Π simultaneously: (1 mark)

$4(1+4\lambda) - 3(-3\lambda) + (2+\lambda) = 19$

$4 + 16\lambda + 9\lambda + 2 + \lambda = 19$

$26\lambda + 6 = 19$

$\lambda = \dfrac{1}{2}$ (1 mark)

Therefore $\overrightarrow{OQ} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + 2 \times \dfrac{1}{2} \times \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$ (1 mark)

$= \begin{pmatrix} 5 \\ -3 \\ 3 \end{pmatrix}$ (1 mark)

b Distance between $P(1,0,2)$ and $Q(5,-3,3)$ is given by

$\sqrt{(5-1)^2 + (-3-0)^2 + (3-2)^2}$ (2 marks)

$= \sqrt{16+9+1}$

$= \sqrt{26}$ (1 mark)

15 a $4(1+6\lambda) + 3(5-2\lambda) - (-3+2\lambda) = 14$ (1 mark)

$4 + 24\lambda + 15 - 6\lambda + 3 - 2\lambda = 14$

$22 + 16\lambda = 14$

$\lambda = -\frac{1}{2}$ (1 mark)

$\boldsymbol{r} = \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 6 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \\ -4 \end{pmatrix}$ (2 marks)

So $P(-2, 6, -4)$.

b $\begin{pmatrix} -2 \\ 6 \\ -4 \end{pmatrix}$ lies on the plane and $\boldsymbol{n} = \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}$ (2 marks)

So distance $= \dfrac{\begin{pmatrix} -2 \\ 6 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}}{\sqrt{4^2 + 3^2 + (-1)^2}}$ (1 mark)

$= \dfrac{-8 + 18 + 4}{\sqrt{26}}$

$= \dfrac{14}{\sqrt{26}}\left(= \dfrac{14\sqrt{26}}{26} = \dfrac{7\sqrt{26}}{13}\right)$ (1 mark)

16 a $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} - \begin{pmatrix} 8 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -6 \\ -2 \\ 6 \end{pmatrix}$ (1 mark)

$\boldsymbol{r} = \begin{pmatrix} 8 \\ 2 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} -6 \\ -2 \\ 6 \end{pmatrix}$ (2 marks)

b $\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = \begin{pmatrix} 12 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \\ -4 \end{pmatrix}$ (1 mark)

$\boldsymbol{r} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} + \mu\begin{pmatrix} 8 \\ -1 \\ 4 \end{pmatrix}$ (2 marks)

c Direction vectors are $\begin{pmatrix} -6 \\ -2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ -1 \\ 4 \end{pmatrix}$

$\begin{pmatrix} -6 \\ -2 \\ 6 \end{pmatrix} \times \begin{pmatrix} 8 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 72 \\ 22 \end{pmatrix}$ (2 marks)

$(8, 2, 0)$ lies on AB and $(4, 4, 4)$ lies on CD

$\overrightarrow{AC} = \begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix}$ (1 mark)

Projection of $\overrightarrow{AC}$ to the vector $\begin{pmatrix} -2 \\ 72 \\ 22 \end{pmatrix}$ is $\dfrac{\left|\begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix} . \begin{pmatrix} -2 \\ 72 \\ 22 \end{pmatrix}\right|}{\sqrt{(-2)^2 + 72^2 + 22^2}}$ (2 marks)

$= \dfrac{8 + 144 + 88}{\sqrt{(-2)^2 + 72^2 + 22^2}}$ (1 mark)

$= \dfrac{240}{\sqrt{5672}}$ (1 mark)

$\left(= \dfrac{240\sqrt{5672}}{5672} = \dfrac{480\sqrt{1418}}{5672} = \dfrac{60\sqrt{1418}}{709} (= 3.19) \right)$

17 Choosing $\lambda = 1$ (say), gives $\boldsymbol{r} = \begin{pmatrix} 10 \\ -4 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 11 \\ -2 \\ 5 \end{pmatrix}$ (1 mark)

Therefore $A(5,8,0), B(10,-4,4)$ and $C(11,-2,5)$ lie on Π (2 marks)

$\overrightarrow{AB} = \begin{pmatrix} 5 \\ -12 \\ 4 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 6 \\ -10 \\ 5 \end{pmatrix}$ (2 marks)

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 5 \\ -12 \\ 4 \end{pmatrix} \times \begin{pmatrix} 6 \\ -10 \\ 5 \end{pmatrix} = \begin{pmatrix} -20 \\ -1 \\ 22 \end{pmatrix}$ (2 marks)

So equation of plane is $\boldsymbol{r} . \begin{pmatrix} -20 \\ -1 \\ 22 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 0 \end{pmatrix} . \begin{pmatrix} -20 \\ -1 \\ 22 \end{pmatrix}$ (2 marks)

$\boldsymbol{r} . \begin{pmatrix} -20 \\ -1 \\ 22 \end{pmatrix} = -108$ (1 mark)

$-20x - y + 22z = -108$ (1 mark)

$\dfrac{20}{108}x + \dfrac{1}{108}y - \dfrac{22}{108}z = 1$ (1 mark)

$\left(\dfrac{5}{27}x + \dfrac{1}{108}y - \dfrac{11}{54}z = 1 \right)$

18 Direction vector of line is $\begin{pmatrix} 2 \\ 5 \\ p \end{pmatrix}$ (1 mark)

Direction normal to plane is $\begin{pmatrix} 5 \\ p \\ p \end{pmatrix}$ (1 mark)

If the angle between the line and the plane is θ , then

$$\sin\theta = \frac{\begin{pmatrix} 2 \\ 5 \\ p \end{pmatrix} \cdot \begin{pmatrix} 5 \\ p \\ p \end{pmatrix}}{\sqrt{2^2+5^2+p^2}\sqrt{5^2+p^2+p^2}}$$ (3 marks)

$$= \frac{10+5p+p^2}{\sqrt{2^2+5^2+p^2}\sqrt{5^2+p^2+p^2}}$$ (1 mark)

θ is maximum when $\sin\theta$ is maximum. (1 mark)

By GDC, maximum occurs when $p = 6.797$ (1 mark)

So maximum value of $\sin\theta$ is 0.96 (1 mark)

$\Rightarrow \theta_{\text{MAX}} = 73.7^\circ$ (1 mark)

19 $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \neq k \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, so L_1 and L_2 are not parallel. (2 marks)

Consider $\boldsymbol{i}$ and $\boldsymbol{j}$ components: (1 mark)

$1+3\lambda = 2+\mu$ and $-\lambda = 1-\mu$ (1 mark)

Solving simultaneously: (1 mark)

$\lambda = 1$, $\mu = 2$ (1 mark)

Substitute into $\boldsymbol{k}$ component: (1 mark)

$2+\lambda = 1+\mu$, $2+1 = 1+2$ (so equations are consistent). (1 mark)

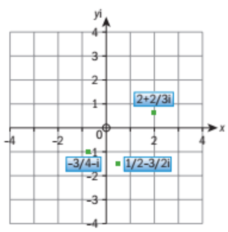
Therefore L_1 and L_2 intersect at the point where $\lambda = 1$ and $\mu = 2$, so are not skew.

(1 mark)

10 Equivalent systems of representation: more complex numbers

Skills check

1



2 $\text{Re}(z_1) = 2, \text{Im}(z_1) = \frac{2}{3}$,

$\text{Re}(z_2) = -\frac{3}{4}, \text{Im}(z_2) = -1$,

$\text{Re}(z_3) = \frac{1}{2}, \text{Im}(z_3) = -\frac{3}{2}$.

3 **a** $1 - 13i$ **b** $-\frac{17}{4} - \frac{7}{4}i$

4 **a** $z^* = 2 + 3i, -z = -2 + 3i$,

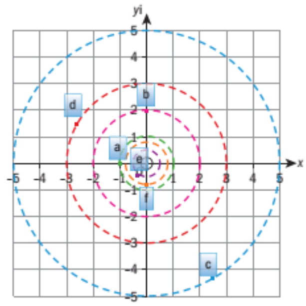
$\frac{1}{z} = \frac{2}{13} + \frac{3}{13}i, |z| = \sqrt{13}$

b $z^* = \frac{4}{5} - \frac{3}{5}i, -z = -\frac{4}{5} - \frac{3}{5}i$,

$\frac{1}{z} = \frac{4}{5} - \frac{3}{5}i, |z| = 1$

Exercise 10A

1



2 **a** $r = \sqrt{(2)^2 + (2)^2} = 2\sqrt{2}$

$\theta = \arctan\left(\frac{2}{2}\right) = \frac{\pi}{4}$

$2 + 2i = 2\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$

b $r = \frac{3}{2}$

$\theta = \frac{\pi}{2}$

$\frac{3}{2}i = \frac{3}{2}\text{cis}\left(\frac{\pi}{2}\right)$

c $r = \sqrt{(-4)^2 + (-3)^2} = 5$

$\theta = \pi + \arctan\left(\frac{3}{4}\right) = 3.78$

$\therefore -4 - 3i = 5\text{cis}(3.78)$

d $r = \sqrt{21^2 + (-20)^2} = 29$

$\theta = 2\pi + \arctan\left(\frac{-20}{21}\right) = 5.52$

$21 - 20i = 29\text{cis}(5.52)$

e $r = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = 2$

$\theta = \pi - \arctan\left(\sqrt{3}\right) = \frac{2\pi}{3}$

$-1 + \sqrt{3}i = 2\text{cis}\left(\frac{2\pi}{3}\right)$

f $-\frac{4}{3}i = \frac{4}{3}\text{cis}\left(\frac{3\pi}{2}\right)$

g $r = \sqrt{\left(\frac{\sqrt{2}}{3}\right)^2 + \left(\frac{\sqrt{2}}{4}\right)^2} = \frac{5\sqrt{2}}{12}$

$\theta = 2\pi + \arctan\left(\frac{-3}{4}\right) = 5.64$

$\therefore \frac{\sqrt{2}}{3} - \frac{\sqrt{2}}{4}i = \frac{5\sqrt{2}}{12}\text{cis}(5.64)$

3 **a** $z = -1$

b $z = 2i$

c $z = \frac{5}{2} - \frac{5\sqrt{3}}{2}i$

d $z = -\frac{3\sqrt{3}}{2} - \frac{3}{2}i$

e $z = -\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4}i$

f $z = -\frac{4}{5}i$

4 **a** $-z = \frac{7}{12}\text{cis}\left(\frac{\pi}{9} + \pi\right) = \frac{7}{12}\text{cis}\left(\frac{10\pi}{9}\right)$

b $z^* = \frac{7}{12}\text{cis}\left(-\frac{\pi}{9}\right)$

c $-z^* = \frac{7}{12}\text{cis}\left(\pi - \frac{\pi}{9}\right) = \frac{7}{12}\text{cis}\left(\frac{8\pi}{9}\right)$

Exercise 10B

1 **a** $z_1z_2 = 8e^{i\left(\frac{\pi}{3}+\frac{\pi}{4}\right)} = 8e^{i\frac{7\pi}{12}}$

b $z_3z_4 = 30\text{cis}\left(90^o + 45^o\right) = 30\text{cis}\left(135^o\right)$

c $z_5z_6 = \frac{5}{9}e^{i\left(\frac{11\pi}{7}+\frac{23\pi}{14}\right)} = \frac{5}{9}e^{i\left(\frac{45}{14}\pi\right)} = \frac{5}{9}e^{i\left(\frac{17}{14}\pi\right)}$ □

d $z_7z_8 = \text{cis}\left(220^o + 275^o\right) = \text{cis}\left(495^o\right) = \text{cis}\left(135^o\right)$

2 **a** $z_1 = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

b $z_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \text{cis}\frac{2\pi}{3} = e^{i\frac{2\pi}{3}}$

c $z_1z_2 = e^{i\frac{3\pi}{4}}e^{i\frac{2\pi}{3}} = e^{i\left(\frac{3\pi}{4}+\frac{2\pi}{3}\right)} = e^{i\left(\frac{17\pi}{12}\right)} = \frac{-\sqrt{6}+\sqrt{2}}{4} - \frac{\sqrt{6}+\sqrt{2}}{4}i$

d $\left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=\left(\frac{\sqrt{2}}{4}-\frac{\sqrt{6}}{4}\right)+i\left(-\frac{\sqrt{2}}{4}-\frac{\sqrt{6}}{4}\right)$

$\cos\frac{17\pi}{12}=\frac{\sqrt{2}-\sqrt{6}}{4};\sin\frac{17\pi}{12}=-\frac{\sqrt{2}+\sqrt{6}}{4}$

e $\tan\frac{17\pi}{12}=\frac{\sin\frac{17\pi}{12}}{\cos\frac{17\pi}{12}}=\frac{\frac{\sqrt{2}+\sqrt{6}}{4}}{\frac{\sqrt{6}-\sqrt{2}}{4}}=\frac{\left(\sqrt{6}+\sqrt{2}\right)^2}{6-2}=\frac{8+2\sqrt{12}}{4}=\frac{8+4\sqrt{3}}{4}=2+\sqrt{3}$

3 a $\sqrt{3}-i=2\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)=2e^{-i\frac{\pi}{6}}$

$z\left(\sqrt{3}-i\right)=2re^{i\left(\theta-\frac{\pi}{6}\right)}$

$2r<3\Rightarrow r<\frac{3}{2}$

$\theta-\frac{\pi}{6}=0\Rightarrow\theta=\frac{\pi}{6}$

$\theta-\frac{\pi}{6}=\pi\Rightarrow\theta=\frac{7\pi}{6}$

$\therefore$ Real if $\theta=\frac{\pi}{6}$ or $\theta=\frac{7\pi}{6}$(up to multiples of π)

and less than 3 if $r<\frac{3}{2}$

b $z\left(-1+i\right)=\sqrt{2}z\left(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)=\sqrt{2}z\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)=\sqrt{2}ze^{i\frac{3\pi}{4}}$

$=\sqrt{2}re^{i\left(\theta+\frac{3\pi}{4}\right)}$

$\theta+\frac{3\pi}{4}=\pi\Rightarrow\theta=\frac{\pi}{4}$

$\theta+\frac{5\pi}{4}=2\pi\Rightarrow\theta=\frac{5\pi}{4}$

$\therefore$ Imaginary for $\theta\neq\frac{\pi}{4}$ or $\theta\neq\frac{5\pi}{4}$ (up to multiples of π)

Modulus greater than 4 if $\left|\sqrt{2}r\right|>4\Rightarrow r>2\sqrt{2}$

4 $\left(\sin\frac{\pi}{12}+i\cos\frac{\pi}{12}\right)\left(\sin\frac{\pi}{6}+i\cos\frac{\pi}{6}\right)\left(\sin\frac{\pi}{4}+i\cos\frac{\pi}{4}\right)$

$i^3\left(\cos\frac{\pi}{12}-i\sin\frac{\pi}{12}\right)\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)=-i\,\mathrm{cis}\left(-\frac{\pi}{12}\right)\bullet\mathrm{cis}\left(-\frac{\pi}{6}\right)\bullet\mathrm{cis}\left(-\frac{\pi}{4}\right)$

$=-ie^{-i\left(\frac{\pi}{12}+\frac{\pi}{6}+\frac{\pi}{4}\right)}=-ie^{-i\frac{\pi}{2}}=\left(-i\right)^2=-1$

Exercise 10C

1 a $\frac{z_1}{z_2}=\frac{3\mathrm{cis}\frac{\pi}{4}}{4\mathrm{cis}\frac{5\pi}{3}}=\frac{3}{4}\mathrm{cis}\left(\frac{\pi}{4}-\frac{5\pi}{3}\right)=\frac{3}{4}\mathrm{cis}\left(-\frac{17\pi}{12}\right)=\frac{3}{4}\mathrm{cis}\left(\frac{7\pi}{12}\right)$

b $\frac{z_3{}^*}{-z_2} = -\frac{5\text{cis}\left(\pi - \frac{7\pi}{6}\right)}{4\text{cis}\left(\pi + \frac{5\pi}{3}\right)} = \frac{5\,\text{cis}\left(\frac{5\pi}{6}\right)}{4\,\text{cis}\left(\frac{8\pi}{3}\right)} = \frac{5}{4}\text{cis}\left(\frac{5\pi}{6} - \frac{2\pi}{3}\right) = \frac{5}{4}\text{cis}\left(\frac{\pi}{6}\right)$

c $\frac{z_1}{z_2 z_3} = \frac{3\text{cis}\,\frac{\pi}{4}}{\left(4\text{cis}\,\frac{5\pi}{3}\right)\left(5\text{cis}\,\frac{7\pi}{6}\right)} = \frac{3}{20}\text{cis}\left(\frac{\pi}{4} - \frac{5\pi}{3} - \frac{7\pi}{6}\right) = \frac{3}{20}\text{cis}\left(-\frac{31\pi}{12}\right) = \frac{3}{20}\text{cis}\left(\frac{17\pi}{12}\right)$

d $\frac{-z_3{}^*}{(z_1 z_2)^*} = -\left(\frac{z_3}{z_1 z_2}\right)^* = -\left(\frac{5}{12}\text{cis}\left(\frac{7\pi}{6} - \frac{\pi}{4} - \frac{5\pi}{3}\right)\right)^* = -\left(\frac{5}{12}\text{cis}\left(-\frac{3\pi}{4}\right)\right)^* =$

$= \left(\frac{5}{12}\text{cis}\left(\pi - \frac{3\pi}{4}\right)\right)^* = \frac{5}{12}\text{cis}\left(-\pi + \frac{3\pi}{4}\right) = \frac{5}{12}\text{cis}\left(-\frac{\pi}{4}\right) = \frac{5}{12}\text{cis}\left(\frac{7\pi}{4}\right)$

2 $1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{2}e^{i\frac{\pi}{4}}$

$1 - \sqrt{3}i = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 2e^{-i\frac{\pi}{3}}$

a $\frac{z_1}{z_2} = \frac{\sqrt{2}e^{i\frac{\pi}{4}}}{2e^{-i\frac{\pi}{3}}} = \frac{\sqrt{2}}{2}\text{e}^{i\left(\frac{\pi}{4}+\frac{\pi}{3}\right)} = \frac{\sqrt{2}}{2}e^{i\frac{7\pi}{12}}$

b $-\frac{z_2{}^*}{z_1} = \sqrt{2}\,\text{e}^{i\left(-\frac{2\pi}{3}-\frac{\pi}{4}\right)} = \sqrt{2}\,\text{e}^{i\left(-\frac{11\pi}{12}\right)}$ or $\sqrt{2}\,\text{e}^{i\left(\frac{13\pi}{12}\right)}$

c $\frac{1}{z_1 z_2} = \frac{1}{\left(\sqrt{2}e^{i\frac{\pi}{4}}\right)\left(2e^{-i\frac{\pi}{3}}\right)} = \frac{1}{2\sqrt{2}e^{i\left(\frac{\pi}{4}-\frac{\pi}{3}\right)}} = \frac{\sqrt{2}}{4}e^{i\frac{\pi}{12}}$

d $-\frac{z_1{}^*}{(z_1 z_2)^*} = -\left(\frac{z_1}{z_1 z_2}\right)^* = -\frac{1}{z_2{}^*} = \frac{1}{2}\text{cis}\,\frac{2\pi}{3} = \frac{1}{2}e^{i\frac{2\pi}{3}}$

3 **a** $\frac{3}{2+2i} = \frac{3}{2\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)} = \frac{3}{2\sqrt{2}\text{cis}\,\frac{\pi}{4}} = \frac{3\sqrt{2}}{4}\text{cis}\left(-\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{4}\text{cis}\left(\frac{7\pi}{4}\right)$

b $\frac{4-4i}{-1+\sqrt{3}i} = \frac{4\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)}{2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)} = \frac{2\sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)}{\text{cis}\left(\frac{2\pi}{3}\right)} = 2\sqrt{2}\text{cis}\left(-\frac{11\pi}{12}\right) = 2\sqrt{2}\,\text{cis}\left(\frac{13\pi}{12}\right)$

c $\frac{\sqrt{15} - \sqrt{5}i}{\sqrt{2} + \sqrt{6}i} = \frac{2\sqrt{5}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)}{2\sqrt{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)} = \frac{\sqrt{10}}{2}\frac{\text{cis}\left(-\frac{\pi}{6}\right)}{\text{cis}\left(\frac{\pi}{3}\right)} = \frac{\sqrt{10}}{2}\text{cis}\left(-\frac{\pi}{2}\right) = \frac{\sqrt{10}}{2}\text{cis}\left(\frac{3\pi}{2}\right)$

4 **a** $z_1 = 5\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \frac{5}{2} + \frac{5\sqrt{3}}{2}i$

b $z_2 = 3 + 3i = 3\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = 3\sqrt{2}\text{cis}\frac{\pi}{4}$

c $\frac{z_1}{z_2} = \frac{5\text{cis}\frac{\pi}{3}}{3\sqrt{2}\text{cis}\frac{\pi}{4}} = \frac{5\sqrt{2}}{6}\text{cis}\frac{\pi}{12}$

$$\frac{z_1}{z_2} = \frac{\frac{5}{2} + \frac{5\sqrt{3}}{2}i}{3 + 3i} = \frac{5}{6}\cdot\frac{1+\sqrt{3}i}{1+i} = \frac{5}{6}\cdot\frac{\left(1+\sqrt{3}i\right)(1-i)}{(1+i)(1-i)}$$
$$= \frac{5}{6}\cdot\frac{1+\sqrt{3}+i\left(\sqrt{3}-1\right)}{2} = \frac{5\left(1+\sqrt{3}\right)}{12} + i\frac{5\left(\sqrt{3}-1\right)}{12}$$

d $\frac{5\sqrt{2}}{6}\cos\frac{\pi}{12} = \frac{5}{12}\left(1+\sqrt{3}\right)$

$$\cos\frac{\pi}{12} = \frac{6}{5\sqrt{2}}\cdot\frac{5}{12}\left(1+\sqrt{3}\right) = \frac{1+\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

e $\frac{5\sqrt{2}}{6}\sin\frac{\pi}{12} = \frac{5\left(\sqrt{3}-1\right)}{12}$

$$\Rightarrow \sin\frac{\pi}{12} = \frac{6}{5\sqrt{2}}\cdot\frac{5\left(\sqrt{3}-1\right)}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

f $\tan\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}} = \frac{\left(\sqrt{6}-\sqrt{2}\right)\left(\sqrt{6}-\sqrt{2}\right)}{6-2} = \frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)^2$

$$= \frac{1}{4}\left(6 - 2\sqrt{12} + 2\right) = \frac{1}{4}\left(8 - 4\sqrt{3}\right) = 2 - \sqrt{3}$$

Exercise 10D

1 $z_1 = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{2}\text{cis}\frac{\pi}{4} = \sqrt{2}e^{i\frac{\pi}{4}}$

a $z_1{}^3 z_2{}^2 = \left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^3\left(2e^{i\frac{2\pi}{3}}\right)^2 = 8\sqrt{2}e^{i\left(\frac{3\pi}{4}+\frac{4\pi}{3}\right)} = 8\sqrt{2}e^{i\frac{25\pi}{12}} = 8\sqrt{2}e^{i\frac{\pi}{12}}$

b $\frac{z_1{}^5}{z_2{}^3} = \frac{4\sqrt{2}e^{i\frac{5\pi}{4}}}{8e^{i2\pi}} = \frac{\sqrt{2}}{2}e^{i\frac{5\pi}{4}}$

c $\left(z_1{}^4\right)^*\left(z_2{}^*\right)^5 = \left(4e^{i\pi}\right)^*\left(2e^{-i\frac{2\pi}{3}}\right)^5 = \left(4e^{-i\pi}\right)\left(32e^{-i\frac{10\pi}{3}}\right) = 128e^{-\frac{13\pi i}{3}} = 128e^{\frac{5\pi i}{3}}$

d $\frac{\left(z_2{}^*\right)^6}{\left(-z_1{}^3\right)^*} = \frac{\left(2e^{-i\frac{2\pi}{3}}\right)^6}{\left(2\sqrt{2}e^{-i\frac{9\pi}{4}}\right)^*} = \frac{64e^{-4i\pi}}{2\sqrt{2}e^{i\frac{9\pi}{4}}} = 16\sqrt{2}e^{-i\frac{25\pi}{4}} = 16\sqrt{2}e^{i\frac{7\pi}{4}}$

2 $z_1 = \sqrt{2}\left(\cos\frac{\pi}{3} - \cos\left(\frac{5\pi}{6}\right)i\right) = \sqrt{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right); z_2 = 2\left(\sin\frac{5\pi}{6} - i\sin\frac{\pi}{3}\right) = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$

$$\frac{z_1^{\,3}}{z_2^{\,5}} = \frac{\left(\sqrt{2}\,e^{i\frac{\pi}{3}}\right)^3}{\left(2\,e^{i\frac{5\pi}{3}}\right)^5} = \frac{2\sqrt{2}}{32}\frac{e^{i\pi}}{e^{i\frac{25\pi}{3}}} = \frac{\sqrt{2}}{16}e^{-i\frac{22\pi}{3}} = \frac{\sqrt{2}}{16}e^{i\frac{2\pi}{3}}$$

3 $\left(\dfrac{\sin\theta + i\cos\theta}{\cos\theta - i\sin\theta}\right)^{2019} = i^{2019}\left(\dfrac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta}\right)^{2019} = i^{2019}$

$= i^{2016}i^3 = \left(i^4\right)^{504}(-i) = -i$

4 **a** $r = \dfrac{1+3i}{2+i} = \dfrac{(1+3i)(2-i)}{(2+i)(2-i)} = \dfrac{5+5i}{5} = 1+i = \sqrt{2}e^{i\frac{\pi}{4}}$

b □ $(2+i)\left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^8 = 16(2+i) = 32+16i$

c $S_9 = (2+i)\dfrac{\left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^9 - 1}{\sqrt{2}e^{i\frac{\pi}{4}} - 1} = (2+i)\dfrac{16\sqrt{2}e^{i\frac{\pi}{4}} - 1}{\sqrt{2}e^{i\frac{\pi}{4}} - 1}$

$= (2+i)\dfrac{\left(16\sqrt{2}e^{i\frac{\pi}{4}} - 1\right)\left(\sqrt{2}e^{-i\frac{\pi}{4}} - 1\right)}{\left(\sqrt{2}e^{-i\frac{\pi}{4}} - 1\right)\left(\sqrt{2}e^{i\frac{\pi}{4}} - 1\right)} = (2+i)\dfrac{32 - 16\sqrt{2}e^{i\frac{\pi}{4}} - \sqrt{2}e^{-i\frac{\pi}{4}} + 1}{2 - \sqrt{2}\left(e^{i\frac{\pi}{4}} + e^{-i\frac{\pi}{4}}\right) + 1}$

$= (2+i)\dfrac{33 - 16\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) - \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)}{3 - \sqrt{2}\left(2\,\frac{1}{\sqrt{2}}\right)}$

$= (2+i)\dfrac{33 - 16(1+i) - (1-i)}{1}$

$= (2+i)(16-15i)$

$= 47 - 14i$

Exercise 10E

1 **a** $\omega^3 + 1 = 0$

$\Rightarrow (\omega+1)(\omega^2 - \omega + 1) = 0$

But $\omega \neq -1$ so it must be the case that $\omega^2 - \omega + 1 = 0$

b $\left(\omega^*\right)^2 - \omega^* + 1 = \left(\omega^2\right)^* - \omega^* + 1^* = \left(\omega^2 - \omega + 1\right)^* = 0^* = 0$

c $\omega^{2019} = \left(\omega^3\right)^{673} = -1$

$\omega^{2019} + \omega^{2020}\left(1 - \omega + \omega^2\right) = -1 + 0 = -1$

2 $\left(1+\omega+\ldots+\omega^{6}\right)(\omega-1)=0 \Rightarrow \omega^{7}-1=0; \omega \neq 1$

$\omega_k = \mathrm{e}^{i\frac{2\pi k}{7}}, k = 1,2,\ldots,6$

$1+\omega+\ldots+\omega^{6}$ can be factorised

$(\omega-\omega_1)(\omega-\omega_2)(\omega-\omega_3)(\omega-\omega_4)(\omega-\omega_5)(\omega-\omega_6)=$

since $\omega_1^{*}=\omega_6, \omega_2^{*}=\omega_5, \omega_3^{*}=\omega_4$

$(\omega-\omega_1)(\omega-\omega_1^{*})(\omega-\omega_2)(\omega-\omega_2^{*})(\omega-\omega_3)(\omega-\omega_3^{*})=$
$\left(\omega^2-2\mathrm{Re}(\omega_1)\omega+1\right)\left(\omega^2-2\mathrm{Re}(\omega_2)\omega+1\right)\left(\omega^2-2\mathrm{Re}(\omega_3)\omega+1\right)=$
$\left(\omega^2-2\cos\frac{2\pi}{7}\omega+1\right)\left(\omega^2-2\cos\frac{4\pi}{7}\omega+1\right)\left(\omega^2-2\cos\frac{6\pi}{7}\omega+1\right)=$
$\left(\omega^2-1.25\omega+1\right)\left(\omega^2+0.445\omega+1\right)\left(\omega^2+1.80\omega+1\right)$

3 a $\left(5^4 e^{i\pi+2ik\pi}\right)^{\frac{1}{4}} = 5e^{i\pi\left(\frac{1}{4}+\frac{k}{2}\right)}$

b $\left(\sqrt{3}-i\right)^{\frac{1}{5}} = 2^{\frac{1}{5}}\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^{\frac{1}{5}} = 2^{\frac{1}{5}}\left(e^{-i\frac{\pi}{6}+2k\pi}\right)^{\frac{1}{5}} = 2^{\frac{1}{5}}e^{i\pi\left(\frac{2k}{5}-\frac{1}{30}\right)}$

c $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{\frac{1}{6}} = \left(e^{i\frac{2\pi}{3}+2k\pi}\right)^{\frac{1}{6}} = e^{i\pi\left(\frac{1}{9}+\frac{k}{3}\right)}$

4 a $z_1 = 1.18+0.334i \Rightarrow \text{Polar}\, 1.22\,\mathrm{e}^{0.277i}$

$z_2 = 1.22\,\mathrm{e}^{i\left(0.277+\frac{\pi}{2}\right)} = -0.334+1.18i$
$z_3 = 1.22\,\mathrm{e}^{i(0.277+\pi)} = -1.18-0.334i$
$z_4 = 1.22\,\mathrm{e}^{i\left(0.277+\frac{3\pi}{2}\right)} = 0.334-1.18i$

b $z_1 = 1.40-0.106i \Rightarrow \text{Polar}\, 1.40\mathrm{e}^{-0.0761i}$

$z_{k+1} = 1.40\mathrm{e}^{i\left(-0.0761+\frac{2\pi k}{5}\right)}, k = 1,2,3,4$
$z_2 = 0.533+1.30i; z_3 = -1.07+0.907i;$
$z_4 = -1.19-0.735i; z_5 = 0.330-1.36i$

c $z_1 = 1.40+0.287i \Rightarrow \text{Polar}\ 1.43\,\mathrm{e}^{0.202i}$

$z_{k+1} = 1.43\,\mathrm{e}^{i\left(0.202i+\frac{k\pi}{3}\right)}, k = 1,2,3,4,5$
$z_2 = 0.452+1.36i; z_3 = -0.949+1.07i;$
$z_4 = -z_1; z_5 = -z_2; z_6 = -z_3$

5 a $8\left(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right) = 8\left(e^{i\frac{3\pi}{4}+2ik\pi}\right)$

b $\operatorname{Re}\left(z^{\frac{1}{6}}\right) = \dfrac{z^{\frac{1}{6}} + (z^*)^{\frac{1}{6}}}{2} = \dfrac{8^{\frac{1}{6}}\left(e^{i\frac{\pi}{8}} + e^{-i\frac{\pi}{8}}\right)}{2} = \sqrt{2}\cos\dfrac{\pi}{8}$

$$\cos\frac{\pi}{4} = 2\cos^2\frac{\pi}{8} - 1 = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \cos\frac{\pi}{8} = \sqrt{\frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)} = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}} = \sqrt{\frac{2+\sqrt{2}}{4}}$$

$$\therefore \operatorname{Re}\left(z^{\frac{1}{6}}\right) = \sqrt{2}\sqrt{\frac{2+\sqrt{2}}{4}} = \sqrt{\frac{2\sqrt{2}+4}{4}} = \frac{\sqrt{2\sqrt{2}+4}}{2}$$

Exercise 10F

1 $P(n):\ (\text{cis}\theta)^n = \text{cis}n\theta$

The statement $P(1)$ is true:
$\text{cis}\theta = \text{cis}\theta$
Assume that $P(k)$ is true for some $k \in \mathbb{Z}^+$
i.e. $(\text{cis}\theta)^k = \text{cis}k\theta$
Then,
$(\text{cis}\theta)^{k+1} = (\text{cis}\theta)^k(\text{cis}\theta) = (\text{cis}k\theta)(\text{cis}\theta)$
$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$
$= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \sin\theta\cos k\theta)$
$= \cos(k\theta + \theta) + i\sin(k\theta + \theta)$ using the compound angle formula
$= \cos((k+1)\theta) + i\sin((k+1)\theta)$
so $P(k) \Rightarrow P(k+1)$

Therefore it has been shown that $P(1)$ is true and that if $P(k)$ is true for some $k \in \mathbb{Z}^+$ then so is $P(k+1)$. Thus, $P(n)$ is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction

2 **a** $z^4 = (\cos\theta + i\sin\theta)^4$

$= \cos^4\theta + (4\cos^3\theta\sin\theta)i - 6\cos^2\theta\sin^2\theta + (-4\cos\theta\sin^3\theta)i + \sin^4\theta$
$= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$

b $z^4 = \cos 4\theta + i\sin 4\theta$

Comparing these with the answers found in part a,

i $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

ii $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

c $\tan 4\alpha = \dfrac{\sin 4\alpha}{\cos 4\alpha} = \dfrac{4\cos^3\alpha\sin\alpha - 4\cos\alpha\sin^3\alpha}{\cos^4\alpha - 6\cos^2\alpha\sin^2\alpha + \sin^4\alpha} = \dfrac{4\dfrac{\sin\alpha}{\cos\alpha} - 4\dfrac{\sin^3\alpha}{\cos^3\alpha}}{1 - 6\dfrac{\sin^2\alpha}{\cos^2\alpha} + \dfrac{\sin^4\alpha}{\cos^4\alpha}} =$

$$= \frac{4\tan\alpha - 4\tan^3\alpha}{1 - 6\tan^2\alpha + \tan^4\alpha}$$

3 a $\left(z + \frac{1}{z}\right)^4 = z^4 + \frac{4z^3}{z} + \frac{6z^2}{z^2} + \frac{4z}{z^3} + \frac{1}{z^4}$

$= z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}$

b $\left(z + \frac{1}{z}\right)^4 = \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$

$= 2\cos 4\theta + 8\cos 2\theta + 6$

Also,

$\left(z + \frac{1}{z}\right)^4 = 16\cos^4\theta$

$\therefore \cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$

c $\therefore \int \cos^4 x dx = \int\left(\frac{1}{8}\cos 4x + \frac{1}{2}\cos 2x + \frac{3}{8}\right)dx$

$= \frac{1}{32}\sin 4x + \frac{1}{4}\sin 2x + \frac{3x}{8} + C$

4 a $\omega^6 - 1 = (\omega^2 - 1)(\omega^4 + \omega^2 + 1) = 0$

$\omega^2 \neq 1$ so it must be that $1 + \omega^2 + \omega^4 = 0$

b $\omega^{102} = (\omega^6)^{17} = 1$

$\omega^{1004} = (\omega^6)^{167}\omega^2 = \omega^2$

$\omega^{20008} = (\omega^6)^{20004}\omega^4 = \omega^4$

$\therefore 1 + \omega^{102} + \omega^{1004} + \omega^{20008} = 1 + (1 + \omega^2 + \omega^4) = 1$

5 Let $z = e^{i\frac{\pi}{8}}$

Let $S = 1 + e^{i\frac{\pi}{8}} + e^{i\frac{\pi}{4}} + \ldots + e^{i\pi}$

$u_1 = 1, r = e^{i\frac{\pi}{8}}$

$$S = \frac{1 - \left(e^{i\frac{\pi}{8}}\right)^9}{1 - e^{i\frac{\pi}{8}}}$$

$$\Rightarrow S = \frac{1 - e^{i\pi}e^{i\frac{\pi}{8}}}{1 - e^{i\frac{\pi}{8}}} = \frac{1 + e^{i\frac{\pi}{8}}}{1 - e^{i\frac{\pi}{8}}} = \frac{\left(1 + e^{i\frac{\pi}{8}}\right)\left(1 - e^{-i\frac{\pi}{8}}\right)}{\left(1 - e^{i\frac{\pi}{8}}\right)\left(1 - e^{-i\frac{\pi}{8}}\right)} = \frac{1 + \left(e^{i\frac{\pi}{8}} - e^{-i\frac{\pi}{8}}\right) - 1}{1 - \left(e^{i\frac{\pi}{8}} + e^{i\frac{\pi}{8}}\right) + 1}$$

$$= \frac{2i\sin\frac{\pi}{8}}{2 - 2\cos\frac{\pi}{8}} = \frac{4i\sin\frac{\pi}{16}\cos\frac{\pi}{16}}{4\sin^2\frac{\pi}{16}} = i\cot\frac{\pi}{16}$$

Chapter review

1 $z = 6e^{-\frac{3\pi}{4}i} = 6\left(\cos\left(\frac{3\pi}{4}\right) - i\sin\left(\frac{3\pi}{4}\right)\right)$

$= 6\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = -3\sqrt{2} - 3\sqrt{2}i$

so $\text{Re}(z) = -3\sqrt{2}, \ \text{Im}(z) = -3\sqrt{2}$

2 $|z_2| = \sqrt{5^2 + (-12)^2} = 13$

$\therefore |z_1{}^2 z_2| = 13r^2 = 52 \Rightarrow r^2 = 4 \Rightarrow r = 2 \ \ (r \geq 0)$

3 $\frac{1}{1+z} = \frac{1+z^*}{(1+z)(1+z^*)} = \frac{1+z^*}{1+(z+z^*)+|z|^2}$

$= \frac{1+z^*}{2+2\text{Re}(z)} = \frac{1}{2}\left(\frac{1+\cos\theta - i\sin\theta}{1+\cos\theta}\right) = \frac{1}{2}\left(1 - i\frac{\sin\theta}{1+\cos\theta}\right)$

$= \frac{1}{2}\left(1 - i\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\right) = \frac{1}{2}\left(1 - i\tan\frac{\theta}{2}\right)$

4 **a** $z^5 = 1 = e^{2in\pi}$

$\Rightarrow z = e^{\frac{2in\pi}{5}} \quad (e.g.\ n = 0, 1, 2, 3, 4)$

$\Rightarrow z = 1, \ z = e^{\frac{2\pi i}{5}}, \ z = e^{\frac{4\pi i}{5}}, \ z = e^{\frac{6\pi i}{5}}, \ z = e^{\frac{8\pi i}{5}}$

b The five roots above can be written as

$1, \ \omega, \ \omega^2, \ \omega^3, \ \omega^4$ i.e. the fifth roots of unity

As a consequence of the fact that the roots of unity sum to zero,

$\text{Re}\left(1 + \omega + \omega^2 + \omega^3 + \omega^4\right) = 0$

$\Rightarrow 1 + \cos\frac{2\pi}{5} + \cos\frac{4\pi}{4} + \cos\frac{6\pi}{5} + \cos\frac{8\pi}{5} = 0$

$\Rightarrow \cos\frac{2\pi}{5} + \cos\frac{4\pi}{4} + \cos\frac{6\pi}{5} + \cos\frac{8\pi}{5} = -1$

5 **a** $z^n + \frac{1}{z^n} = (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$

$= (\cos n\theta + i\sin n\theta) + (\cos(-n\theta) + i\sin(-n\theta))$

$= (\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta)$

$= 2\cos n\theta$

b $\left(z + \frac{1}{z}\right)^6 = z^6 + \frac{6z^5}{z} + \frac{15z^4}{z^2} + \frac{20z^3}{z^3} + \frac{15z^2}{z^4} + \frac{6z}{z^5} + \frac{1}{z^6}$

$= z^6 + 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}$

c Using part **a**,

$$\left(z+\frac{1}{z}\right)^6=\left(z^6+\frac{1}{z^6}\right)+6\left(z^4+\frac{1}{z^4}\right)+15\left(z^2+\frac{1}{z^2}\right)+20$$
$$=2\cos 6\theta+12\cos 4\theta+30\cos 2\theta+20$$

but also

$$\left(z+\frac{1}{z}\right)^6=(2\cos\theta)^6=64\cos^6\theta$$

$$\therefore 64\cos^6\theta=2\cos 6\theta+12\cos 4\theta+30\cos 2\theta+20$$
$$\Rightarrow \cos^6\theta=\frac{1}{32}\cos 6\theta+\frac{3}{16}\cos 4\theta+\frac{15}{32}\cos 2\theta+\frac{5}{16}$$

d $$\therefore \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos^6 x\,dx=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\frac{1}{32}\cos 6x+\frac{3}{16}\cos 4x+\frac{15}{32}\cos 2x+\frac{5}{16}\right)dx$$

$$=\left[\frac{1}{192}\sin 6x+\frac{3}{64}\sin 4x+\frac{15}{64}\sin 2x+\frac{5x}{16}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$
$$=\frac{5\pi}{32}-\left(-\frac{1}{192}+\frac{15}{64}+\frac{5\pi}{64}\right)$$
$$=\frac{5\pi}{64}-\frac{11}{48}$$

6 $\theta=-\arctan\left(\frac{41}{23}\right)=-1.0595656...=-1.06\ \ (3\text{s.f.})$

7 $3z_1-2z_2+\frac{1}{2}z_3$

$$=\left(9-10\cos\frac{17\pi}{83}+\sqrt{2}\cos\left(-\frac{\pi}{4}\right)\right)+i\left(3-10\sin\frac{17\pi}{83}+\sqrt{2}\sin\left(-\frac{\pi}{4}\right)\right)$$
$$=1.9997458...-(3.9996610....)i$$
$$=2.00-(4.00)i\qquad(\text{to 3s.f.})$$

8 $\lim_{x\to 0}\frac{\sin 4x}{\sin x}=\lim_{x\to 0}\frac{4\cos 4x}{\cos x}=4$ using L'Hopital's Rule

9 The distance from the centre to each vertex is 2

Therefore split the pentagon into five isosceles triangles, and using the formula

Triangle area $=\frac{1}{2}ab\sin C$, we have

Pentagon area $=\frac{5}{2}(2)^2\left(\sin\frac{2\pi}{5}\right)=9.51056...=9.51$ to 2d.p.

10 $\omega^2=(a+2i)^2=a^2+4ai-4=(a^2-4)+4ai$

$$\arg(\omega^2)=\arctan\left(\frac{4a}{a^2-4}\right)=1$$
$$\frac{4a}{a^2-4}=\tan(1)$$
$$a^2\tan(1)-4a-4\tan(1)=0$$
$$a=3.66$$

Exam-style questions

11 a $z_1 z_2 = 4\,\text{cis}\left(-\frac{\pi}{3}\right) 3\,\text{cis}\left(\frac{5\pi}{6}\right) = 12\,\text{cis}\left(\frac{5\pi}{6} - \frac{\pi}{3}\right)$ (1 mark)

$= 12\,\text{cis}\left(\frac{\pi}{2}\right)$ (1 mark)

$= 12\text{i}$ (1 mark)

b $\frac{z_1}{z_2} = \frac{4\,\text{cis}\left(-\frac{\pi}{3}\right)}{3\,\text{cis}\left(\frac{5\pi}{6}\right)} = \frac{4}{3}\text{cis}\left(-\frac{\pi}{3} - \frac{5\pi}{6}\right)$ (1 mark)

$= \frac{4}{3}\text{cis}\left(-\frac{7\pi}{6}\right) = \frac{4}{3}\text{cis}\left(\frac{5\pi}{6}\right)$ (1 mark)

So $\left(\frac{z_1}{z_2}\right)^3 = \left(\frac{4}{3}\text{cis}\left(\frac{5\pi}{6}\right)\right)^3 = \frac{64}{27}\text{cis}\left(\frac{15\pi}{6}\right)$ (1 mark)

$= \frac{64}{27}\text{cis}\left(\frac{\pi}{2}\right)$ (1 mark)

$= \frac{64}{27}\text{i}$ (1 mark)

c $z_1^{\,2} = 16\,\text{cis}\left(-\frac{2\pi}{3}\right)$

So $\left(z_1^{\,2}\right)^* = 16\,\text{cis}\left(\frac{2\pi}{3}\right)$ (1 mark)

$= 16\left(\cos\frac{2\pi}{3} + \text{i}\sin\frac{2\pi}{3}\right)$

$= 16\left(-\frac{1}{2} + \text{i}\frac{\sqrt{3}}{2}\right)$ (1 mark)

$= -8 + 8\sqrt{3}\text{i}$ (1 mark)

12 $|1+\text{i}| = \sqrt{2}$ (1 mark)

$\arg(1+\text{i}) = \frac{\pi}{4}$ (1 mark)

$1 + \text{i} = \sqrt{2}\,\text{cis}\frac{\pi}{4}$

$(1+\text{i})^{10} = \left(\sqrt{2}\right)^{10}\text{cis}\frac{10\pi}{4}$ by de Moivre's theorem (1 mark)

$= 2^5\,\text{cis}\frac{5\pi}{2}$ (1 mark)

$= 2^5 \operatorname{cis} \frac{\pi}{2}$

$= 32\mathrm{i}$ (1 mark)

13 a $|z| = \sqrt{1 + \left(\sqrt{3}\right)^2} = 2$ (1 mark)

$\arg z = -\frac{\pi}{3}$ (1 mark)

$z = 2\operatorname{cis}\left(-\frac{\pi}{3}\right)$ (1 mark)

b $z^n = 2^n \operatorname{cis}\left(-\frac{n\pi}{3}\right)$ (1 mark)

$z^n \in \mathbb{R} \Rightarrow -\frac{n\pi}{3} = 2\pi k$ (1 mark)

So $n = 6$ (1 mark)

c $\left(1 - \mathrm{i}\sqrt{3}\right)^{15} = 2^{15} \operatorname{cis}\left(-\frac{15\pi}{3}\right)$ (1 mark)

$= 2^{15} \operatorname{cis}(-5\pi)$

$= 2^{15} \operatorname{cis}(\pi)$ (1 mark)

$= -2^{15} \ (= -32768)$ (1 mark)

14 a $(\cos\theta + \mathrm{i}\sin\theta)^5 = \cos^5\theta + 5\cos^4\theta(\mathrm{i}\sin\theta) + 10\cos^3\theta(\mathrm{i}\sin\theta)^2$

$+10\cos^2\theta(\mathrm{i}\sin\theta)^3 + 5\cos\theta(\mathrm{i}\sin\theta)^4 + (\mathrm{i}\sin\theta)^5$ (2 marks)

$= \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10\mathrm{i}\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + \mathrm{i}\sin^5\theta$ (1 mark)

b By de Moivre's theorem, $(\cos\theta + \mathrm{i}\sin\theta)^5 = \cos 5\theta + \mathrm{i}\sin 5\theta$ (1 mark)

Equating real parts of each expression: (1 mark)

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\left(1 - \cos^2\theta\right) + 5\cos\theta\left(1 - \cos^2\theta\right)^2$ (1 mark)

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta\left(1 - 2\cos^2\theta + \cos^4\theta\right)$ (1 mark)

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

15 a Let $z^3 = -27\mathrm{i}$

$\left|z^3\right| = 27$ (1 mark)

$\arg\left(z^3\right) = -\frac{\pi}{2}$ (1 mark)

$z^3 = 27\left(\cos\left(-\frac{\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{2}\right)\right)$ (1 mark)

$$z^3 = 27\left(\cos\left(-\frac{\pi}{2}+2\pi k\right)+\mathrm{i}\sin\left(-\frac{\pi}{2}+2\pi k\right)\right)$$ (1 mark)

$$z^3 = 27\left(\cos\left(\frac{4\pi k-\pi}{2}\right)+\mathrm{i}\sin\left(\frac{4\pi k-\pi}{2}\right)\right)$$

$$z = 3\left(\cos\left(\frac{4\pi k-\pi}{6}\right)+\mathrm{i}\sin\left(\frac{4\pi k-\pi}{6}\right)\right)$$ (1 mark)

Choosing $k = 1,2,3$ (or equivalent)

$$z_1 = 3\operatorname{cis}\frac{\pi}{2}$$ (1 mark)

$$z_2 = 3\operatorname{cis}\frac{7\pi}{6}$$ (1 mark)

$$z_3 = 3\operatorname{cis}\frac{11\pi}{6}$$ (1 mark)

b Area $= 3\times\left(\frac{1}{2}\times 3\times 3\times\sin\frac{2\pi}{3}\right)$ (2 marks)

$$= 3\times\left(\frac{1}{2}\times 3\times 3\times\frac{\sqrt{3}}{2}\right)$$

$$= \frac{27\sqrt{3}}{4}$$ (1 mark)

16 a $z^n = (\cos\theta+\mathrm{i}\sin\theta)^n = \cos n\theta+\mathrm{i}\sin n\theta$ (1 mark)

$$\frac{1}{z^n} = z^{-n} = (\cos\theta+\mathrm{i}\sin\theta)^{-n} = \cos(-n\theta)+\mathrm{i}\sin(-n\theta) = \cos n\theta-\mathrm{i}\sin n\theta$$ (2 marks)

So $z^n+\frac{1}{z^n} = (\cos n\theta+\mathrm{i}\sin n\theta)+(\cos n\theta-\mathrm{i}\sin n\theta)$ (1 mark)

$$= 2\cos n\theta$$

b $\left(z+\frac{1}{z}\right)^4 = z^4+4z^3\left(\frac{1}{z}\right)+6z^2\left(\frac{1}{z}\right)^2+4z\left(\frac{1}{z}\right)^3+\left(\frac{1}{z}\right)^4$ (2 marks)

$$= z^4+4z^2+6+\frac{4}{z^2}+\frac{1}{z^4}$$ (1 mark)

$$= z^4+\frac{1}{z^4}+4\left(z^2+\frac{1}{z^2}\right)+6$$ (1 mark)

$$= 2\cos 4\theta+4(2\cos 2\theta)+6$$ (1 mark)

$$= 2\cos 4\theta+8\cos 2\theta+6$$

Now $\left(z+\frac{1}{z}\right)^4 = (2\cos\theta)^4 = 16\cos^4\theta$ (1 mark)

Therefore $\cos^4\theta \equiv \frac{1}{16}(2\cos 4\theta+8\cos 2\theta+6)$

c $\int_0^{\frac{\pi}{6}} \cos^4\theta \, d\theta = \frac{1}{16}\int_0^{\frac{\pi}{6}} (2\cos 4\theta + 8\cos 2\theta + 6) \, d\theta$ (1 mark)

$= \frac{1}{16}\left[\frac{1}{2}\sin 4\theta + 4\sin 2\theta + 6\theta\right]_0^{\frac{\pi}{6}}$ (1 mark)

$= \frac{1}{16}\left[\frac{1}{2}\sin\frac{2\pi}{3} + 4\sin\frac{\pi}{3} + \pi\right]_0^{\frac{\pi}{6}}$ (1 mark)

$= \frac{1}{16}\left(\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) + 4\left(\frac{\sqrt{3}}{2}\right) + \pi\right)$ (1 mark)

$= \frac{1}{16}\left(\frac{\sqrt{3}}{4} + 2\sqrt{3} + \pi\right)$

$= \frac{1}{16}\left(\frac{9\sqrt{3}}{4} + \pi\right)$ (1 mark)

$= \frac{\pi}{16} + \frac{9\sqrt{3}}{64}$

17 a $\left|\frac{\sqrt{3}+i}{\sqrt{3}-i}\right| = \frac{\left|\sqrt{3}+i\right|}{\left|\sqrt{3}-i\right|} = \frac{2}{2} = 1$ (2 marks)

So $r = 1$

$\arg\left(\frac{\sqrt{3}+i}{\sqrt{3}-i}\right) = \arg\left(\sqrt{3}+i\right) - \arg\left(\sqrt{3}-i\right)$ (1 mark)

$= \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}$ (1 mark)

So $\theta = \frac{\pi}{3}$ (1 mark)

$\frac{\sqrt{3}+i}{\sqrt{3}-i} = e^{i\frac{\pi}{3}}$

b i $\left(\sqrt{3}+i\right)^n + \left(\sqrt{3}-i\right)^n = \left(2\operatorname{cis}\frac{\pi}{6}\right)^n + \left(2\operatorname{cis}\left(-\frac{\pi}{6}\right)\right)^n$ (1 mark)

$= 2^n \operatorname{cis}\frac{n\pi}{6} + 2^n \operatorname{cis}\left(-\frac{n\pi}{6}\right)$ (1 mark)

$= 2^n\left(\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6} + \cos\left(-\frac{n\pi}{6}\right) + i\sin\left(-\frac{n\pi}{6}\right)\right)$

$= 2^n\left(\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6} + \cos\frac{n\pi}{6} - i\sin\frac{n\pi}{6}\right)$ (1 mark)

$= 2^n\left(\cos\frac{n\pi}{6} + \cos\frac{n\pi}{6}\right)$ (1 mark)

$= 2^n\left(2\cos\dfrac{n\pi}{6}\right)$ (1 mark)

$= 2^{n+1}\cos\left(\dfrac{n\pi}{6}\right)$

ii $\left(\sqrt{3}+\mathrm{i}\right)^8+\left(\sqrt{3}-\mathrm{i}\right)^8 = 2^9\cos\left(\dfrac{8\pi}{6}\right)$ (1 mark)

$= 2^9\cos\left(\dfrac{4\pi}{3}\right)$

$= 2^9\left(-\dfrac{1}{2}\right)$ (1 mark)

$= -2^8 = -256$ (1 mark)

18 a $\omega^* = \omega^2$ (1 mark)

$\left(1+\omega+\omega^*\right)^2 = \left(1+\omega+\omega^2\right)^2$

$= \left(\dfrac{1-\omega^3}{1-\omega}\right)^2$ (1 mark)

$= 0^2 = 0$ (1 mark)

b $\left(1+\omega+3\omega^2\right)^2 = \left(1+\omega+\omega^2+2\omega^2\right)^2$ (1 mark)

$= \left(2\omega^2\right)^2 = 4\omega^4$ (1 mark)

$= 4\omega$ (1 mark)

c $\left(1+2\omega+3\omega^2\right)\left(1+3\omega+2\omega^2\right) = \left(1+\omega+\omega^2+\omega+2\omega^2\right)\left(1+\omega+\omega^2+2\omega+\omega^2\right)$ (1 mark)

$= \left(\omega+2\omega^2\right)\left(2\omega+\omega^2\right)$ since $1+\omega+\omega^2 = 0$ (1 mark)

$= 2\omega^2+\omega^3+4\omega^3+2\omega^4$ (1 mark)

$= 2\omega^2+5\omega^3+2\omega^4$

$= 2\omega^2+5+2\omega$ (1 mark)

$= 2\left(1+\omega+\omega^2\right)+3$ (1 mark)

$= 2\times 0+3$

$= 3$ (1 mark)

19 $\mathrm{i} = \cos\dfrac{\pi}{2}+\mathrm{i}\sin\dfrac{\pi}{2}$ (1 mark)

$= \cos\left(\dfrac{\pi}{2}+2\pi k\right)+\mathrm{i}\sin\left(\dfrac{\pi}{2}+2\pi k\right)$ (1 mark)

$$= \cos\left(\frac{4\pi k + \pi}{2}\right) + \mathrm{i}\sin\left(\frac{4\pi k + \pi}{2}\right)$$

So $z - 2\mathrm{i} = \cos\left(\frac{4\pi k + \pi}{6}\right) + \mathrm{i}\sin\left(\frac{4\pi k + \pi}{6}\right)$ (1 mark)

$k = 0 \Rightarrow z - 2\mathrm{i} = \cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}$ (1 mark)

$k = 1 \Rightarrow z - 2\mathrm{i} = \cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}$ (1 mark)

$k = 2 \Rightarrow z - 2\mathrm{i} = \cos\frac{9\pi}{6} + \mathrm{i}\sin\frac{9\pi}{6}$ (1 mark)

$z - 2\mathrm{i} = \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$

$z - 2\mathrm{i} = -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$

$z - 2\mathrm{i} = -\mathrm{i}$ (3 marks)

So roots are $z_1 = \frac{\sqrt{3}}{2} + \frac{5}{2}\mathrm{i}$, $z_2 = -\frac{\sqrt{3}}{2} + \frac{5}{2}\mathrm{i}$ and $z_3 = \mathrm{i}$ (1 mark)

11 Valid comparisons and informed decisions: probability distributions

Skills Check

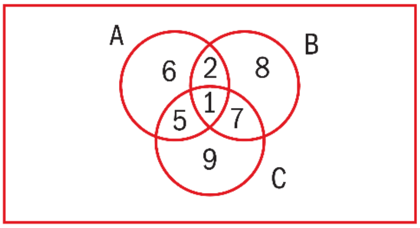
1 Fred wants to sit at either end, thus there are two possible positions for him, there are 4! ways to arrange the remaining four friends, thus there are $2 \times 4! = 48$ different seating arrangements

2

	1	1	3	4	6	8
1	2	2	4	5	7	9
1	2	2	4	5	7	9
3	4	4	6	7	9	11
4	5	5	7	8	10	12
6	7	7	9	10	12	14
8	9	9	11	12	14	16

Counting the even outcomes (highlighted in yellow), there are 18 possible outcomes, so the probability is $P(\text{even}) = \frac{18}{36} = \frac{1}{2}$.

3 a



b $P(\text{all three languages}) = \frac{1}{38}$

4 a $P(\text{History}) = \frac{17}{29}$

b $P(\text{not Physics}) = \frac{15}{29}$

c $P(\text{not Physics} \mid \text{not History}) = \frac{3}{12} = \frac{1}{4}$

Exercise 11A

1 **a** $P(\text{French and German}) = P(\text{French}) + P(\text{German}) - P(\text{French or German})$

$$= \frac{10}{20} + \frac{8}{20} - \left(1 - \frac{5}{20}\right) = \frac{3}{20}$$

b $P(\text{exactly one language}) = 1 - P(\text{both}) - P(\text{neither})$

$$= 1 - \frac{3}{20} - \frac{5}{20} = \frac{3}{5}$$

2 Number of ways of picking 4 letters: $\binom{9}{4} = \frac{9!}{4!(9-4)!} = \frac{362880}{24 \times 120} = 126$

$P(\text{at least one vowel}) = 1 - P(\text{no vowels}) = 1 - \frac{\binom{6}{4}}{126} = \frac{37}{42}$

3 Number of ways of picking 2 fruits: $\binom{9}{2} = \frac{9!}{2!(9-2)!} = \frac{362880}{2 \times 5040} = 36$

a $P(\text{2 kiwis}) = \frac{\binom{3}{2}}{36} = \frac{1}{12}$

b $P(\text{two different fruits}) = \frac{6 \times 3}{36} = \frac{1}{2}$

4 Number of ways of picking 3 crayons: $\binom{12}{3} = \frac{12!}{3!(12-3)!} = \frac{479001600}{6 \times 362880} = 220$

a $P(\text{all green}) = \frac{\binom{7}{3}}{220} = \frac{7}{44}$

b $P(\text{not all same colour}) = 1 - P(\text{all same colour}) = 1 - P(\text{all green}) - P(\text{all blue})$

$$= 1 - \frac{7}{44} - \frac{\binom{5}{3}}{220} = \frac{35}{44}$$

5 **a** $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.5 - 0.2 = 0.6$

b $P(B \cap A') = P(B) - P(A \cap B) = 0.5 - 0.2 = 0.3$

c $P(A' \cap B') = P(A') - P(A' \cap B) = 1 - 0.3 - 0.3 = 0.4$

6 **a** $P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.4 + 0.6 - 0.7 = 0.3$

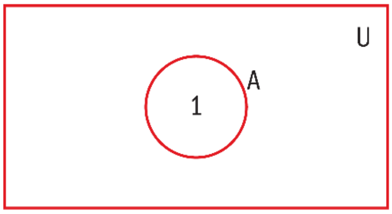
b $P(A \cap B') = P(A) - P(A \cap B) = 0.4 - 0.3 = 0.1$

c $P(A' \cup B') = P(A') + P(B') - P(A' \cap B')$
$P(A' \cap B') = P(B') - P(A \cap B') = 0.4 - 0.1 = 0.3$

so $P(A' \cup B') = 0.6 + 0.4 - 0.3 = 0.7$

7 $U = \{1, 2, 3\}$, $A = \{3\}$, $B = \varnothing$

a



b **i** $P(A) = \frac{n(A)}{n(U)} = \frac{1}{3}$

ii $P(B) = \frac{n(B)}{n(U)} = \frac{0}{3} = 0$

Exercise 11B

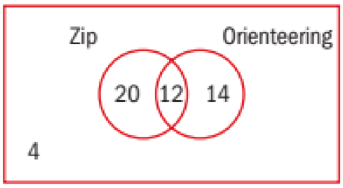
1 **a** Independent

b Independent

c Not independent

d Independent

2 **a**



b **i** P(at least one student went zip lining) = 1, as 32 students chose zip lining.

ii $P(\text{orienteering} \mid \text{zip lining}) = \frac{P(\text{both})}{P(\text{zip lining})}$,

$P(\text{both}) = P(\text{zip lining}) + P(\text{orienteering}) - P(\text{either}) = \frac{32}{50} + \frac{26}{50} - \frac{46}{50} = \frac{6}{25}$,

$P(\text{orienteering} \mid \text{zip lining}) = \frac{\frac{6}{25}}{\frac{32}{50}} = \frac{3}{8}$

iii $P(\text{orienteering}) = \frac{26}{50} = \frac{13}{25}$

iv $P(\text{zip lining} \mid \text{orienteering}) = \frac{P(\text{both})}{P(\text{orienteering})} = \frac{\frac{6}{25}}{\frac{13}{25}} = \frac{6}{13}$

3 **a** $P(\text{art}) = \frac{5+2+7+3}{35} = \frac{17}{35}$

b $P(\text{biology}) = \frac{4+2+7+6}{35} = \frac{19}{35}$

c $P(\text{chemistry}) = \frac{1+3+7+6}{35} = \frac{17}{35}$

d $P(\text{art} \mid \text{biology}) = \dfrac{2+7}{2+7+4+6} = \dfrac{9}{19}$

e $P(\text{biology} \mid \text{chemistry}) = \dfrac{7+6}{3+7+6+1} = \dfrac{13}{17}$

f $P(\text{chemistry} \mid \text{not art}) = \dfrac{6+1}{35-(5+2+3+7)} = \dfrac{7}{18}$

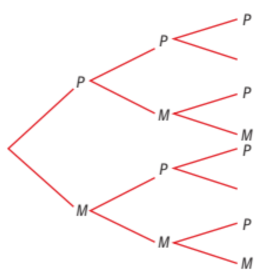
g $P(\text{homework} \mid \text{no homework}) = 0$

4 $P(\text{both blue} \mid \text{same colour}) = \dfrac{P(\text{both blue} \cap \text{same colour})}{P(\text{same colour})}$

$$= \frac{P(\text{both blue})}{P(\text{same colour})} = \frac{P(\text{both blue})}{P(\text{both blue}) + P(\text{both yellow})}$$

$$= \frac{\frac{3}{8}\times\frac{2}{7}}{\frac{3}{8}\times\frac{2}{7}+\frac{5}{8}\times\frac{4}{7}} = \frac{3}{13}$$

5 a



b i $P(\text{all puzzles}) = \dfrac{8}{13}\times\dfrac{7}{12}\times\dfrac{6}{11} = \dfrac{28}{143}$

ii $P(\text{at least one puzzle}) = 1 - P(\text{no puzzles}) = 1 - \left(\dfrac{5}{13}\times\dfrac{4}{12}\times\dfrac{3}{11}\right) = \dfrac{138}{143}$

6 Let A be the event that he scores on the first shot and B be the event that he scores on the second shot. Now, we have $P(A) = 0.85$, $P(B' \mid A) = 0.1$ and $P(B \mid A') = 0.75$. We want to find

$$\begin{aligned} P((A \cap B') \cup (A' \cap B)) &= P(A \cap B') + P(A' \cap B) \\ &= P(A)P(B' \mid A) + P(A')P(B \mid A') \\ &= 0.85 \times 0.1 + 0.15 \times 0.75 = 0.1975 \end{aligned}$$

7 a $P(B') = 1 - P(B) = 1 - (P(A)P(B \mid A) + P(A')P(B \mid A')) = 1 - \left(\dfrac{1}{3}\times\dfrac{3}{5}+\dfrac{2}{3}\times\dfrac{1}{2}\right) = \dfrac{7}{15}$

b $P(A' \cup B') = P((A \cap B)') = 1 - P(A \cap B) = 1 - P(A)P(B \mid A) = \dfrac{4}{5}$

Exercise 11C

1 $P(\text{yellow}) = P(\text{both yellow})P(\text{yellow} \mid \text{both yellow}) + P(\text{one of each})P(\text{yellow} \mid \text{one of each})$
$+P(\text{both green})P(\text{yellow} \mid \text{both green})$

$$= \frac{\binom{5}{2}}{\binom{13}{2}} \times \frac{4}{10} + \frac{\binom{5}{1}\binom{8}{1}}{\binom{13}{2}} \times \frac{3}{10} + \frac{\binom{8}{2}}{\binom{13}{2}} \times \frac{2}{10} = \frac{18}{65}$$

$$P(\text{yellow from } A \mid \text{yellow}) = \frac{\binom{5}{2}}{\binom{13}{2}} \times \frac{2}{4} + \frac{\binom{5}{1}\binom{8}{1}}{\binom{13}{2}} \times \frac{1}{3} + \frac{\binom{8}{2}}{\binom{13}{2}} \times 0 = \frac{55}{234}$$

2 **a** $P(\text{male}) = P(\text{high income})P(\text{male} \mid \text{high}) + P(\text{medium income})P(\text{male} \mid \text{medium})$
$+P(\text{low income})P(\text{male} \mid \text{low})$
$= (0.1 \times 0.5) + (0.65 \times 0.7) + (0.25 \times 0.8) = 0.705$

b $P(\text{high income} \mid \text{male})$

$= P(\text{high income and male}) \div P(\text{male})$
$= 0.05 \div 0.705$
$= 0.07$

c $P(\text{female} \mid \text{high income}) = 0.5,$ from the question.

3 Let D be the event that the transistor is defective and M_i be the event that the transistor is from machine i, now:

$$P(M_2 \mid D') = \frac{P(M_2)P(D' \mid M_2)}{P(M_1)P(D' \mid M_1) + P(M_2)P(D' \mid M_2) + P(M_3)P(D' \mid M_3)}$$
$$= \frac{0.45 \times 0.97}{0.5 \times 0.96 + 0.45 \times 0.97 + 0.05 \times 0.92}$$
$$= 0.4535$$

4 Let A_1 be the event that the teacher took two 320GB laptops, A_2 be the event that the teacher took one 160GB laptop and one 320GB laptop and A_3 be the event that the teacher took two 160GB laptops. Also let B be the event that the student took a 160GB laptop. We wish to find

$$P(A_1 \mid B) = \frac{P(A_1)P(B \mid A_1)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + P(A_3)P(B \mid A_3)}$$

$$= \frac{\frac{\binom{8}{2}}{\binom{20}{2}} \times \frac{12}{18}}{\frac{\binom{8}{2}}{\binom{20}{2}} \times \frac{12}{18} + \frac{\binom{12}{1}\binom{8}{1}}{\binom{20}{2}} \times \frac{11}{18} + \frac{\binom{12}{2}}{\binom{20}{2}} \times \frac{10}{18}} = \frac{\frac{14}{95} \times \frac{12}{18}}{\frac{14}{95} \times \frac{12}{18} + \frac{48}{95} \times \frac{11}{18} + \frac{33}{95} \times \frac{10}{18}}$$

$$= \frac{28}{171}$$

5 $P(\text{basket}) = \frac{4}{12} \times 0.009 + \frac{4}{12} \times 0.006 + \frac{4}{12} \times 0.002 = 0.00567 = 0.567\%$

6 **a** $P(\text{Rh+}) = (0.45 \times 0.84) + (0.37 \times 0.84) + (0.14 \times 0.83) + (0.04 \times 0.75) = 0.835 = 83.5\%$

b $P(\text{AB} \mid \text{Rh-}) = \dfrac{P(\text{AB})P(\text{Rh-} \mid \text{AB})}{P(\text{O})P(\text{Rh-} \mid \text{O}) + P(\text{A})P(\text{Rh-} \mid \text{A}) + P(\text{B})P(\text{Rh-} \mid \text{B}) + P(\text{AB})P(\text{Rh-} \mid \text{AB})}$

$= \dfrac{0.04 \times 0.25}{(0.45 \times 0.16) + (0.37 \times 0.16) + (0.14 \times 0.17) + (0.04 \times 0.25)}$

$= 0.0606 = 6.06\%$

Exercise 11D

1 a Yes

b No because the sum of the probabilities is not equal to 1

2 a

x	0	1	2	3	4
$P(X = x)$	0	$\frac{1}{2}$	4	$\frac{27}{2}$	32

Not a probability distribution as some of the probabilities are greater than 1

b

x	0	1	2	3	4
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{6}{17}$	$\frac{12}{11}$

Not a probability distribution as one of the probabilities is greater than 1

3 a Need to find k such that $k(10+11+12) = 1$, so $k = \dfrac{1}{10+11+12} = \dfrac{1}{33}$

b The mode is the most likely outcome, which in this case is $x = 12$

c $P(X \text{ is even}) = P(X = 10) + P(X = 12) = \dfrac{10}{33} + \dfrac{12}{33} = \dfrac{2}{3}$

Exercise 11E

1 $E(X) = \Sigma x P(X = x) = 0 \times \dfrac{1}{6} + 1 \times \dfrac{1}{6} + 2 \times \dfrac{1}{3} + 3 \times \dfrac{1}{4} + 4 \times \dfrac{1}{12} = \dfrac{23}{12}$

2 Have to find k such that $0.1 + 0.2 + k + 2k + (k - 0.1) = 1$

so $1 = 4k - 0.2 \Rightarrow 4k = 0.8 \Rightarrow k = 0.2$,

$E(X) = \Sigma x P(X = x) = (4 \times 0.1) + (5 \times 0.2) + (6 \times 0.2) + (7 \times 0.4) + (8 \times 0.1) = 6.2$

3 $P(X = 3) = 1 - P(X \le 2) = 0.15$, $P(X = 2) = P(X \le 2) - P(X \le 1) = 0.35$, let $p_0 = P(X = 0)$ and $p_1 = P(X = 1)$. Then $E(X) = 0 \times p_0 + 1 \times p_1 + 2 \times 0.35 + 3 \times 0.15 = 1.45$ gives us that $p_1 = 0.3$ and therefore $p_0 = 0.2$.

a

x	0	1	2	3
$P(X = x)$	0.2	0.3	0.35	0.15

b The mode is $x = 2$

4 There are $\binom{10}{3} = 120$ ways of picking the three students.

X	0	1	2	3
$P(X = x)$	$\frac{1}{120}$	$\frac{1}{30}$	$\frac{1}{20}$	$\frac{1}{30}$

5 **a** Need to find c such that $0 + 2c + c + 3c^2 + \left(3c^2 + c\right) + c = 1$, so $c = -1$ or $c = \frac{1}{6}$, as probabilities must be greater than or equal to 0, it must be $c = \frac{1}{6}$

b $E(X) = \Sigma x P(X = x) = 1(0) + 2\left(\frac{1}{3}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{12}\right) + 5\left(\frac{1}{4}\right) + 6\left(\frac{1}{6}\right) = \frac{15}{4} = 3.75$

Exercise 11F

1 **a** $E(X) = \Sigma x P(X = x) = 1 \times \frac{1}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{2} = \frac{1}{8} + \frac{6}{8} + \frac{3}{2} = \frac{19}{8} = 2.375$

b $E(5X) = 5E(X) = 5 \times 2.375 = 11.875$

c $E(X^2) = 1^2 \times \frac{1}{8} + 2^2 \times \frac{3}{8} + 3^2 \times \frac{1}{2} = \frac{1}{8} + \frac{12}{8} + \frac{9}{2} = \frac{49}{8} = 6.125$

d $Var(X) = E(X^2) - (E(X))^2 = 6.125 - 2.375^2 = 0.484$ (3.s.f.)

e Standard deviation of $X = \sqrt{Var(X)}$

$= \sqrt{0.484\ldots} = 0.696$ (3s.f.)

2 **a** $P(X \geq 1) = 2P(X \leq 2) \Rightarrow a + a + 2a + b = 2(a + a + a) \Rightarrow b = 2a$, so combining with $5a + b = 1$ gives us that $7a = 1 \Rightarrow a = \frac{1}{7}$ and $b = \frac{2}{7}$

b $E(X) = 0 \times \frac{1}{7} + 1 \times \frac{1}{7} + 2 \times \frac{1}{7} + 3 \times \frac{2}{7} + 4 \times \frac{2}{7} = \frac{17}{7}$

$E(X^2) = 0^2 \times \frac{1}{7} + 1^2 \times \frac{1}{7} + 2^2 \times \frac{1}{7} + 3^2 \times \frac{2}{7} + 4^2 \times \frac{2}{7} = \frac{55}{7}$

c $Var(X) = E(X^2) - (E(X))^2 = \frac{55}{7} - \left(\frac{17}{7}\right)^2 = \frac{96}{49}$

3 Let X_1 be the value of the bottom card and let X_2 be the value of the top card

a $P(S = 4) = P(X_1 = A \cap X_2 = 3) + P(X_1 = 3 \cap X_2 = A) = \frac{1}{10} \times \frac{1}{9} + \frac{1}{10} \times \frac{1}{9} = \frac{2}{90} = \frac{1}{45}$

$$P(S = 8) = P(X_1 = A \cap X_2 = 7) + P(X_1 = 2 \cap X_2 = 6) + P(X_1 = 3 \cap X_2 = 5) +$$
$$+ P(X_1 = 5 \cap X_2 = 3) + P(X_1 = 6 \cap X_2 = 2) + P(X_1 = 7 \cap X_2 = A) =$$
$$= 6 \times \left(\frac{1}{10} \times \frac{1}{9}\right) = \frac{6}{90} = \frac{1}{15}$$

$$P(S=11)=P(X_1=A\cap X_2=10)+P(X_1=2\cap X_2=9)+P(X_1=3\cap X_2=8)+$$
$$+P(X_1=4\cap X_2=7)+P(X_1=5\cap X_2=6)+P(X_1=6\cap X_2=7)+P(X_1=7\cap X_2=4)+$$
$$+P(X_1=8\cap X_2=3)+P(X_1=9\cap X_2=2)+P(X_1=10\cap X_2=A)=$$
$$=10\times\left(\frac{1}{10}\times\frac{1}{9}\right)=\frac{10}{90}=\frac{1}{9}$$

b

s	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$P(S=s)$	$\frac{2}{90}$	$\frac{2}{90}$	$\frac{4}{90}$	$\frac{4}{90}$	$\frac{6}{90}$	$\frac{6}{90}$	$\frac{8}{90}$	$\frac{8}{90}$	$\frac{10}{90}$	$\frac{8}{90}$	$\frac{8}{90}$	$\frac{6}{90}$	$\frac{6}{90}$	$\frac{4}{90}$	$\frac{4}{90}$	$\frac{2}{90}$	$\frac{2}{90}$

c $E(S)=11$ (because of symmetry)

$$E\left(S^2\right)=\Sigma s^2P(S=s)-E(S)^2=\frac{407}{3}-121=\frac{44}{3}$$

4 a Need to find k such that $k\left((8-4)^2+(8-5)^2+(8-6)^2+(8-7)^2\right)+k\left(1^2+2^2+3^2\right)=1$, so

$$30k+14k=1\Rightarrow k=\frac{1}{30+14}=\frac{1}{44}$$

b $P(T=4)=\frac{1}{44}(8-4)^2=\frac{4}{11}$

$$P(T\le 4)=\frac{1}{44}\left(1^2+2^2+3^2\right)+\frac{4}{11}=\frac{15}{22}$$

$$P(T=4\mid T\le 4)=\frac{P(T=4\cap T\le 4)}{P(T\le 4)}=\frac{\frac{4}{11}}{\frac{15}{22}}=\frac{8}{15}$$

c $E(T)=1\times\frac{1^2}{44}+2\times\frac{2^2}{44}+3\times\frac{3^2}{44}+4\times\frac{(8-4)^2}{44}+5\times\frac{(8-5)^2}{44}+6\times\frac{(8-6)^2}{44}+7\times\frac{(8-7)^2}{44}$

$=4$

$Var(T)=E\left(T^2\right)-(E(T))^2$

$$=1^2\times\frac{1^2}{44}+2^2\times\frac{2^2}{44}+3^2\times\frac{3^2}{44}+4^2\times\frac{(8-4)^2}{44}+5^2\times\frac{(8-5)^2}{44}+6^2\times\frac{(8-6)^2}{44}+7^2\times\frac{(8-7)^2}{44}-16$$

$$=\frac{193}{11}-16=\frac{17}{11}$$

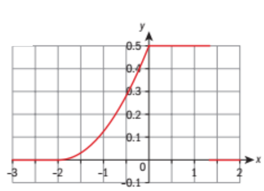
d The mode is the most likely value which is $t=4$

Exercise 11G

1 a Need to find k such that $1=\int_{-2}^{0}k(x+2)^2\,dx+\int_{0}^{\frac{4}{3}}4k\,dx$,

$$1=\int_{-2}^{0}k(x+2)^2\,dx+\int_{0}^{\frac{4}{3}}4k\,dx=\left[\frac{k}{3}(x+2)^3\right]_{-2}^{0}+\left[4kx\right]_{0}^{\frac{4}{3}}=\frac{8k}{3}+\frac{16k}{3}=8k\Rightarrow k=\frac{1}{8}$$

b

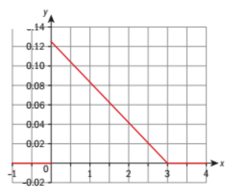


c **i** $P(X \le -1) = \int_{-2}^{-1} \frac{(x+2)^2}{8}\,\mathrm{d}x = \left[\frac{(x+2)^3}{24}\right]_{-2}^{-1} = \frac{1}{24}$

ii $P(0 \le X \le 1) = \int_{0}^{1} \frac{1}{2}\,\mathrm{d}x = \left[\frac{x}{2}\right]_{0}^{1} = \frac{1}{2}$

iii $P(-1 \le X \le 1) = P(-1 \le X \le 0) + \frac{1}{2} = \int_{-1}^{0} \frac{(x+2)^2}{8}\,\mathrm{d}x + \frac{1}{2} = \left[\frac{(x+2)^3}{24}\right]_{-1}^{0} + \frac{1}{2} = \frac{7}{24} + \frac{1}{2} = \frac{19}{24}$

2 **a**



b Need to find k such that $1 = \int_{0}^{3} k(3-x)\,\mathrm{d}x$, $1 = \int_{0}^{3} k(3-x)\,\mathrm{d}x = \left[-k\left(\frac{x^2}{2} - 3x\right)\right]_{0}^{3} = \frac{9k}{2} \Rightarrow k = \frac{2}{9}$

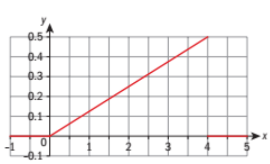
c $P(1.2 \le X \le 2.3) = \int_{1.2}^{2.3} \frac{2(3-x)}{9}\,\mathrm{d}x = \left[\frac{6x - x^2}{9}\right]_{1.2}^{2.3} = \frac{11}{36}$

3 **a** Need to find c such that $1 = \int_{0}^{1} 2x^c\,\mathrm{d}x$, $1 = \int_{0}^{1} 2x^c\,\mathrm{d}x = \left[\frac{2x^{c+1}}{c+1}\right]_{0}^{1} = \frac{2}{c+1} \Rightarrow c = 1$

b $P(X < 0.5) = \int_{0}^{0.5} 2x\,\mathrm{d}x = \left[x^2\right]_{0}^{0.5} = \frac{1}{4}$

Exercise 11H

1



a This function is a probability density function because it is non-negative for all possible values and the integral is equal to 1. $\int_{0}^{4} \frac{x}{8}\,\mathrm{d}x = \left[\frac{x^2}{16}\right]_{0}^{4} = 1$

b $P(1 < x < 20) = \int_{1}^{4} \frac{x}{8}\,\mathrm{d}x + \int_{4}^{20} 0\,\mathrm{d}x = \left[\frac{x^2}{16}\right]_{1}^{4} + 0 = \frac{15}{16}$

c Mean $= \int_{0}^{4} \frac{x^2}{8}\,\mathrm{d}x = \left[\frac{x^3}{24}\right]_{0}^{4} = \frac{8}{3}$

Mode: As $f(x)$ is strictly increasing, the mode occurs at the point $x = 4$

Median: Need to find M such that $0.5 = \int_0^M \frac{x}{8}\,dx$, so

$$0.5 = \int_0^M \frac{x}{8}\,dx = \left[\frac{x^2}{16}\right]_0^M = \frac{M^2}{16} \Rightarrow M = \pm 2\sqrt{2} \Rightarrow M = 2\sqrt{2}$$

Standard deviation $= \sqrt{\int_0^4 \frac{x^3}{8}\,dx - \left(\frac{8}{3}\right)^2} = \sqrt{\left[\frac{x^4}{32}\right]_0^4 - \left(\frac{8}{3}\right)^2} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$

2 Need to find a such that $1 = \int_0^2 ax(2-x)\,dx$, so $1 = \int_0^2 ax(2-x)\,dx = \left[a\left(x^2 - \frac{x^3}{3}\right)\right]_0^2 = \frac{4a}{3} \Rightarrow a = \frac{3}{4}$

a $E(X) = \int_0^2 \frac{3}{4}x^2(2-x)\,dx = \left[\frac{8x^3 - 3x^4}{16}\right]_0^2 = 1$

b $Var(X) = \int_0^2 \frac{3}{4}x^3(2-x)\,dx - 1 = \left[\frac{3(5x^4 - 2x^5)}{40}\right]_0^2 - 1 = \frac{6}{5} - 1 = \frac{1}{5}$

c Need to find M such that $0.5 = \int_0^M \frac{3}{4}x(2-x)\,dx$, so

$$0.5 = \int_0^M \frac{3}{4}x(2-x)\,dx = \left[\frac{3}{4}\left(x^2 - \frac{x^3}{3}\right)\right]_0^M = \frac{3M^2}{4} - \frac{M^3}{4} \Rightarrow M = 1, 1 \pm \sqrt{3} \Rightarrow M = 1$$

d As $f(x)$ is symmetric, the mode occurs at the middle value $x = 1$

3 a $P\left(\frac{\pi}{12} < X < \frac{\pi}{6}\right) = \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} 2\cos(2x)\,dx = [\sin(2x)]_{\frac{\pi}{12}}^{\frac{\pi}{6}} = \frac{1}{2}\left(\sqrt{3} - 1\right)$

b Median: Need to find M such that $0.5 = \int_0^M 2\cos(2x)\,dx$, so

$$0.5 = \int_0^M 2\cos(2x)\,dx = [\sin(2x)]_0^M = \sin(2M) \Rightarrow M = \frac{\sin^{-1}(0.5)}{2} = \frac{\pi}{12}$$

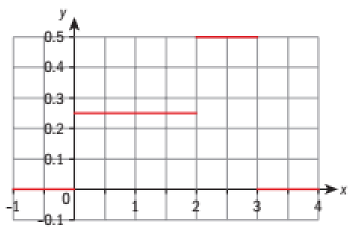
Mean $= \int_0^{\frac{\pi}{4}} 2x\cos(2x)\,dx = \left[\frac{1}{2}\cos(2x) + x\sin(2x)\right]_0^{\frac{\pi}{4}} = \frac{1}{4}(\pi - 2)$

Mode: As $f(x)$ is decreasing over the whole interval (from knowledge about cos) we know that the mode is at $x = 0$

4 a Need to find k such that $1 = \int_0^2 k\,dx + \int_2^3 2k\,dx$, so

$$1 = \int_0^2 k\,dx + \int_2^3 2k\,dx = [kx]_0^2 + [2kx]_2^3 = 2k + 2k = 4k \Rightarrow k = \frac{1}{4}$$

b Median is $x = 1$.



c $E(X)=\int_0^2 \frac{x}{4}\,dx+\int_2^3 \frac{x}{2}\,dx=\left[\frac{x^2}{8}\right]_0^2+\left[\frac{x^2}{4}\right]_2^3=\frac{1}{2}+\frac{5}{4}=\frac{7}{4}$

$Var(X)=\int_0^2 \frac{x^2}{4}\,dx+\int_2^3 \frac{x^2}{2}\,dx-\left(\frac{7}{4}\right)^2=\left[\frac{x^3}{12}\right]_0^2+\left[\frac{x^3}{6}\right]_2^3-\frac{49}{16}=\frac{2}{3}+\frac{19}{6}-\frac{49}{16}=\frac{37}{48}$

5 **a** Need to find b in terms of a such that $1=\int_0^3 ax^2+b\,dx$, so

$1=\int_0^3 ax^2+b\,dx=\left[\frac{ax^3}{3}+bx\right]_0^3=9a+3b\Rightarrow b=\frac{1}{3}-3a$

b $0.5=\int_0^1 ax^2-3a+\frac{1}{3}\,dx=\left[\frac{1}{3}\left(x-9ax+ax^3\right)\right]_0^1=\frac{1}{3}-\frac{8a}{3}\Rightarrow a=-\frac{1}{16}$

c $E(X)=\int_0^3 \frac{25x}{48}-\frac{x^3}{16}\,dx=\left[\frac{1}{48}\left(\frac{25x^2}{2}-\frac{3x^4}{4}\right)\right]_0^3=\frac{69}{64}$

$Var(X)=\int_0^3 \frac{25x^2}{48}-\frac{x^4}{16}\,dx-\left(\frac{69}{64}\right)^2=\left[\frac{1}{48}\left(\frac{25x^3}{3}-\frac{3x^5}{5}\right)\right]_0^3-\frac{4761}{4096}=\frac{33}{20}-\frac{4761}{4096}=\frac{9987}{20480}$

Exercise 11I

1 **a** $P(X=3)=\binom{8}{3}(0.25)^3(0.75)^5=0.208$

b $P(X<10)=1$

c $P(X=0\text{ or }1)=\binom{8}{0}(0.25)^0(0.75)^0+\binom{8}{2}(0.25)^1(0.75)^1=0.367$

2 **a** $E(X)=np=6\times0.4=2.4$

b $Var(X)=np(1-p)=6\times0.4\times0.6=1.44$

c Computing $P(X=0)=0.0467$, $P(X=1)=0.187$, $P(X=2)=0.311$, $P(X=3)=0.276$, $P(X=4)=0.138$, $P(X=5)=0.0369$ and $P(X=6)=0.0410$ shows that 2 is the most likely value

3 **a** $P(X=5)=\frac{5!}{5!0!}\times0.5^5\times(1-0.5)^0=0.5^5=\left(\frac{1}{5}\right)^5=\frac{1}{32}$

b

x	0	1	2	3	4	5
$P(X=x)$	0.0313	0.1563	0.3125	0.3125	0.1563	0.0313

c $E(X)=np=5\times0.5=2.5$

d $Var(X)=np(1-p)=5\times0.5\times0.5=1.25$

e From the table, the modal values are 2 and 3

4 $X \sim \mathrm{B}(7, 0.85)$

a $P(X=1) = \binom{7}{1} \times 0.85^1 \times 0.15^6 = 0.00007$

b $P(X>2) = 1 - P(X \le 2) = 1 - \left(\binom{7}{0}0.85^0 \times 0.15^7 + \binom{7}{1}0.85^1 \times 0.15^6 + \binom{7}{2}0.85^2 \times 0.15^5\right) = 0.9988$

c $P(X \ge 3) = P(X > 2) = 0.9988$ (from part **b**)

d $P(X \le 4) = 1 - P(X > 4) = 1 - \left(\binom{7}{5}0.85^5 \times 0.15^2 + \binom{7}{6}0.85^6 \times 0.15^1 + \binom{7}{7}0.85^7 \times 0.15^0\right) = 0.0738$

5 $X \sim \mathrm{B}\left(12, \frac{2}{3}\right)$

a $E(X) = np = 12 \times \frac{2}{3} = 8$ so it is expected that he will pass 8 and therefore fail 4

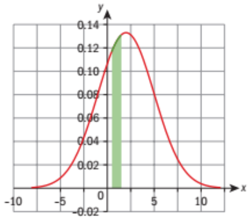
b $P(X>6) = \binom{12}{7}\left(\frac{2}{3}\right)^7\left(\frac{1}{3}\right)^5 + \cdots + \binom{12}{12}\left(\frac{2}{3}\right)^{12}\left(\frac{1}{3}\right)^0 = 0.8223$

c Computing the probabilities close to the mean $P(X=7) = 0.191$, $P(X=8) = 0.238$ and $P(X=9) = 0.213$ shows that 8 is the most likely value

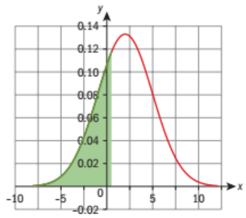
Exercise 11J

1 **a** $P(0.5 < X < 1.5) = P\left(\frac{0.5-2}{3} < X < \frac{1.5-2}{3}\right) = \Phi(-0.1667) - \Phi(-0.5) = 0.4338 - 0.3085$

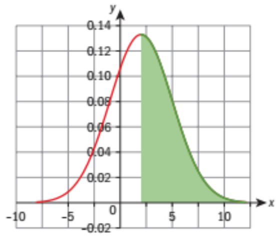
$= 0.1253$



b $P(X < 0.5) = P\left(X < \frac{0.5-2}{3}\right) = \Phi(-0.5) = 0.3085$



c $P(X \geq 2) = P\left(X \geq \frac{2-2}{3}\right) = 1 - \Phi(0) = 0.5$



2 a $P(X < 25) = P\left(X < \frac{25-30}{8}\right) = \Phi(-0.625) = 0.2660$

b $P(17 \leq X < 35) = P\left(\frac{17-30}{8} \leq X < \frac{35-30}{8}\right) = \Phi(0.625) - \Phi(-1.625) = 0.7340 - 0.0521$

$= 0.6819$

c $P(X \geq 12) = P\left(X \geq \frac{12-30}{8}\right) = 1 - \Phi(-2.25) = 1 - 0.5 = 1 - 0.0122 = 0.9878$

3 a Mean: 5, Standard deviation: 3

b $P(X < 4) = P\left(Z < \frac{4-5}{3}\right) = \Phi(-0.3333) = 0.3695$

c $0.011 = P(X < a) = P\left(Z < \frac{a-5}{3}\right) = \Phi\left(\frac{a-5}{3}\right) \Rightarrow \Phi^{-1}(0.011) = -2.2904 = \frac{a-5}{3} \Rightarrow a = -1.8712$

$0.871 = P(X \geq b) = P\left(Z \geq \frac{b-5}{3}\right) = 1 - \Phi\left(\frac{b-5}{3}\right) \Rightarrow \frac{b-5}{3} = \Phi^{-1}(1-0.871) = -1.1311$

$\Rightarrow b = 1.6067$

4 a Mean $= \frac{220 \times 11 + 240 \times 21 + 260 \times 38 + 280 \times 17 + 300 \times 13}{11 + 21 + 38 + 17 + 13} = \frac{26000}{100} = 260$

Standard deviation

$= \sqrt{\frac{220^2 \times 11 + 240^2 \times 21 + 260^2 \times 38 + 280^2 \times 17 + 300^2 \times 13}{11 + 21 + 38 + 17 + 13} - 260^2} = \sqrt{\frac{6813600}{100} - 67600}$

$= 23.152$

b $\Phi^{-1}(0.05) = -1.6449$ and $\Phi^{-1}(0.95) = 1.6449$, so need to find a and b such that $\frac{a-260}{23.152} = -1.6449$ and $\frac{b-260}{23.152} = 1.6449$. Therefore $a = 221.9$ g and $b = 298.1$ g

5 $X \sim N(150, 0.5)$

a $P(X < 149) = P\left(Z < \frac{149-150}{0.5}\right) = \Phi(-2) = 0.0228$

b $P(X > 151.5) = P\left(Z > \frac{151.5-150}{0.5}\right) = 1 - \Phi(3) = 0.00135$

c $P(149 < X < 151) = P\left(\frac{149-150}{0.5} < Z < \frac{151-150}{0.5}\right) = \Phi(2) - \Phi(-2) = 0.9773 - 0.0228 = 0.9545$

6 $T \sim N(13.2, 1.5)$

a $P(T < 12.1) = P\left(Z < \frac{12.1 - 13.2}{1.5}\right) = \Phi(-0.7333) = 0.232$

$P(T > 14.9) = P\left(Z > \frac{14.9 - 13.2}{1.5}\right) = 1 - \Phi(1.1333) = 1 - 0.871 = 0.129$

b If $P(T < t) = 0.444$ then $\frac{t - 13.2}{1.5} = \Phi^{-1}(0.444) = -0.1408 \Rightarrow t = 12.989$

Exercise 11K

1 $0.321 = P(X \le 4) = P\left(Z < \frac{4-8}{\sigma}\right) = \Phi\left(\frac{4-8}{\sigma}\right) \Rightarrow \Phi^{-1}(0.321) = -0.4649 = \frac{4-8}{\sigma}$

$\Rightarrow \sigma = 8.604 \Rightarrow \sigma^2 = 74.03$

2 $0.345 = P(X \le 1) = P\left(Z < \frac{1-\mu}{\sigma}\right) = \Phi\left(\frac{1-\mu}{\sigma}\right) \Rightarrow \Phi^{-1}(0.345) = -0.3989 = \frac{1-\mu}{\sigma}$ and

$0.943 = P(X \le 3) = P\left(Z < \frac{3-\mu}{\sigma}\right) = \Phi\left(\frac{3-\mu}{\sigma}\right) \Rightarrow \Phi^{-1}(0.943) = 1.5805 = \frac{3-\mu}{\sigma}$ solving simultaneously gives $\mu = 1.403$ and $\sigma = 1.010$

3 $0.013 = P(X \le 1) = P\left(Z < \frac{1-1.02}{\sigma}\right) = \Phi\left(\frac{1-1.02}{\sigma}\right) \Rightarrow \Phi^{-1}(0.013) = -2.2262 = \frac{1-1.02}{\sigma}$

$\Rightarrow \sigma = 0.00898$

4 **a** $0.203 = P(X > 46.8) = P\left(Z > \frac{46.8-\mu}{\sigma}\right) = 1 - \Phi\left(\frac{46.8-\mu}{\sigma}\right) \Rightarrow \Phi^{-1}(1-0.203) = 0.8310$

$= \frac{46.8-\mu}{\sigma}$ and $0.315 = P(X \le 42.6) = P\left(Z < \frac{42.6-\mu}{\sigma}\right) = \Phi\left(\frac{42.6-\mu}{\sigma}\right) \Rightarrow \Phi^{-1}(0.315)$

$= -0.4817 = \frac{42.6-\mu}{\sigma}$ solving simultaneously gives $\mu = 44.141$ and $\sigma = 3.200$

b $P\left(|X-\mu| < \frac{\sigma}{2}\right) = P\left(\mu - \frac{\sigma}{2} < X < \mu + \frac{\sigma}{2}\right) = P\left(\frac{-1}{2} < Z < \frac{1}{2}\right) = \Phi\left(\frac{1}{2}\right) - \Phi\left(\frac{-1}{2}\right)$

$= 0.6915 - 0.3085 = 0.383$

5 **a** $P(200 < X < 350) = P\left(\frac{200-320}{20} < Z < \frac{350-320}{20}\right) = \Phi\left(\frac{350-320}{20}\right) - \Phi\left(\frac{200-320}{20}\right)$

$= 0.9332 - 0 = 0.9332$

b $0.1 = P(X > m) = P\left(Z > \frac{m-320}{20}\right) = 1 - \Phi\left(\frac{m-320}{20}\right) \Rightarrow \Phi^{-1}(1-0.1) = 1.2816 = \frac{m-320}{20}$

$\Rightarrow m = 345.63\text{g}$

c $P(X > 350) = P\left(Z > \frac{350-320}{20}\right) = 1 - \Phi\left(\frac{350-320}{20}\right) = 0.0668$, the expected number sold that weighed more than 350g is $500 \times 0.0668 = 33.4$

d $0.15 = P(Y > 400) = P\left(Z > \frac{400-\mu}{\sigma}\right) = 1 - \Phi\left(\frac{400-\mu}{\sigma}\right) \Rightarrow \Phi^{-1}(1-0.15) = 1.0364 = \frac{400-\mu}{\sigma}$ and

$0.1 = P(X \le 370) = P\left(Z < \frac{370-\mu}{\sigma}\right) = \Phi\left(\frac{370-\mu}{\sigma}\right) \Rightarrow \Phi^{-1}(0.1) = -1.2816 = \frac{370-\mu}{\sigma}$ solving simultaneously gives $\mu = 386.59$ g and $\sigma = 12.942$ g

6 We wish to find the maximal value of σ such that

$0.01 < P(X < 1) = P\left(Z < \frac{1-1.03}{\sigma}\right) = \Phi\left(\frac{1-1.03}{\sigma}\right) \Rightarrow \Phi^{-1}(0.01) = -2.3264 < \frac{1-1.03}{\sigma} \Rightarrow \sigma < 0.0129$

Chapter review

1 $U = \{Mon, Tue, Wed, Thu, Fri, Sat, Sun\}$, $A = \{(Mon, Mon)\}$,

$B = \{(Mon, Mon), (Tue, Tue), (Wed, Wed), (Thu, Thu), (Fri, Fri), (Sat, Sat), (Sun, Sun)\}$,

$C = \{(Mon, Tue), (Tue, Wed), (Wed, Thu), (Thu, Fri), (Fri, Sat), (Sat, Sun), (Sun, Mon),$
$(Mon, Sun), (Sun, Sat), (Sat, Fri), (Fri, Thu), (Thu, Wed), (Wed, Tue), (Tue, Mon)\}$

a $P(A) = \frac{n(A)}{n(U)} = \frac{1}{49}$

b $P(B) = \frac{n(B)}{n(U)} = \frac{7}{49} = \frac{1}{7}$

c $P(C) = \frac{n(C)}{n(U)} = \frac{14}{49} = \frac{2}{7}$

2 $U = \{0,1,2,3,4,5,6,7,8,9\}$, $A = \{3,6,7,9\}$, $P(A) = \frac{n(A)}{n(U)} = \frac{4}{10} = \frac{2}{5}$

3 Want to find n such that $P(\text{six at least once}) > \frac{9}{10} \Rightarrow P(\text{no sixes}) = \frac{1}{10}$, so

$\frac{1}{10} \le \binom{n}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^n = 1 \times 1 \times \left(\frac{5}{6}\right)^n \Rightarrow n \ge \frac{\log(10)}{\log\left(\frac{6}{5}\right)} = 12.629\ldots \Rightarrow n = 13$

4 **a** $P(\text{first red}) = \frac{7}{10}$

b $P(\text{second red}) = P(\{RR, WR\}) = \frac{7}{10} \times \frac{6}{9} + \frac{3}{10} \times \frac{7}{9} = \frac{7}{10}$

c $P(\text{exactly one red}) = P(\{RW, WR\}) = \frac{7}{10} \times \frac{3}{9} + \frac{3}{10} \times \frac{7}{9} = \frac{7}{15}$

d $P(\text{at least one red}) = P(\{RW, WR, RR\}) = \frac{7}{10} \times \frac{3}{9} + \frac{3}{10} \times \frac{7}{9} + \frac{7}{10} \times \frac{6}{9} = \frac{14}{15}$

5 **a** $P(\text{infection}) = 0.2 \times 0.1 + 0.8 \times 0.75 = 0.62$

b $P(\text{vaccinated} \mid \text{infection}) = \frac{P(\text{vaccinated} \cap \text{infection})}{P(\text{infection})} = \frac{0.1}{0.62} = 0.161$

c $P(\text{not vaccinated} \mid \text{infection}) = 1 - P(\text{vaccinated} \mid \text{infection}) = 1 - 0.161 = 0.839$

6 $P(\text{2 wins}) = \binom{8}{8}\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^0 = \frac{1}{256}$,

$P(\text{1 win}) = \binom{8}{7}\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^1 + \binom{8}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^3 + \binom{8}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^5 + \binom{8}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^7 = \frac{1}{2}$,

$P(\text{loss}) = 1 - \frac{1}{256} - \frac{1}{2} = \frac{127}{256}$. By encoding a win as a 1 and a loss as a -1, the expected number of wins and losses is $2 \times \frac{1}{256} + 1 \times \frac{1}{2} + (-1) \times \frac{127}{256} = \frac{3}{256}$

7 a $P(\text{no defective}) = \binom{5}{0}\left(\frac{4}{20}\right)^0\left(\frac{16}{20}\right)^5 = 0.328$

b $P(\text{at least 3 defective}) = P(\text{3 defective}) + P(\text{4 defective}) + P(\text{5 defective}) = \sum_{i=3}^{5}\binom{5}{i}\left(\frac{4}{20}\right)^i\left(\frac{16}{20}\right)^{5-i}$

$= 0.0579$

8 a $P(H > 1500) = P\left(Z > \frac{1500-1300}{125}\right) = 1 - \Phi\left(\frac{1500-1300}{125}\right) = 0.0548$

b $P(H < 1050) = P\left(Z < \frac{1050-1300}{125}\right) = \Phi\left(\frac{1050-1300}{125}\right) = 0.0228 = 2.28\%$

c $\left(P(1200 < H < 1400)\right)^2 = \left(P\left(\frac{1200-1300}{125} < H < \frac{1400-1300}{125}\right)\right)^2$

$= \left(\Phi\left(\frac{1400-1300}{125}\right) - \Phi\left(\frac{1200-1300}{125}\right)\right)^2 = (0.7881 - 0.2119)^2 = 0.332$

9 $0.04 = P(X > 2.01) = P\left(Z > \frac{2.01-\mu}{\sigma}\right) = 1 - \Phi\left(\frac{2.01-\mu}{\sigma}\right) \Rightarrow \Phi^{-1}(1-0.04) = 1.7507 = \frac{2.01-\mu}{\sigma}$ and

$0.07 = P(X \le 1.99) = P\left(Z < \frac{1.99-\mu}{\sigma}\right) = \Phi\left(\frac{1.99-\mu}{\sigma}\right) \Rightarrow \Phi^{-1}(0.07) = -1.4758 = \frac{1.99-\mu}{\sigma}$ solving simultaneously gives $\mu = 2.00$ and $\sigma = 0.00620$

10 a $P(\text{faulty}) = 0.01 \times 0.35 + 0.03 \times 0.20 + 0.025 \times 0.24 + 0.02 \times 0.21 = 0.0197$

b $P(\text{conveyor } D \mid \text{faulty}) = \frac{P(\text{conveyor } D \cap \text{faulty})}{P(\text{faulty})} = \frac{0.02 \times 0.21}{0.0197} = 0.213$

11 To show that $k = \pi$, have to show that $\int_0^1 \pi x \sin(\pi x)\,dx = 1$,

$\int_0^1 \pi x \sin(\pi x)\,dx = \left[\frac{\sin(\pi x)}{\pi} - x\cos(\pi x)\right]_0^1 = 1 - 0 = 1$.

$\text{Mean} = \int_0^1 \pi x^2 \sin(\pi x)\,dx = \left[\frac{(2-\pi^2x^2)\cos(\pi x) + 2\pi x\sin(\pi x)}{\pi^2}\right]_0^1 = 1 - \frac{4}{\pi^2}$

Variance

$= \int_0^1 \pi x^3 sin(\pi x)\,dx - \left(1 - \frac{4}{\pi^2}\right)^2$

$= \left[\frac{\pi x(6-\pi^2x^2)\cos(\pi x) + 3(\pi^2x^2-2)\sin(\pi x)}{\pi^3}\right]_0^1 - \left(1-\frac{4}{\pi^2}\right)^2 = \left(1-\frac{6}{\pi^2}\right) - \left(1-\frac{4}{\pi^2}\right)^2$

$= \frac{2(\pi^2-8)}{\pi^4}$

Exam-style questions

12 a If they were mutually exclusive, then $P(A\cap B)=0$, (1 mark)

but since they are independent, we have $P(A\cap B)=P(A)P(B)=0.3\times 0.8\neq 0$.

Therefore, we have a contradiction, and so A and B are not mutually exclusive.

(1 mark)

b **i** $P(A\cap B)=P(A)P(B)=0.3\times 0.8=0.24$ (2 marks)

ii $P(A\cup B)=P(A)+P(B)-P(A\cap B)=0.3+0.8-0.24=0.86$ (2 marks)

iii $P(A\mid B')=\dfrac{P(A\cap B')}{P(B')}=\dfrac{P(A)-P(A\cap B)}{P(B')}=\dfrac{0.3-0.24}{0.2}=\dfrac{0.06}{0.2}=0.3$ (2 marks)

iv $P(A'\cap B)=P(B)-P(A\cap B)=0.8-0.24=0.56$ (2 marks)

13 a $\dfrac{k}{2}+k+k^2+2k^2+\dfrac{k}{2}=1$ (1 mark)

$3k^2+2k-1=0$ (1 mark)

$(3k-1)(k+1)=0$ (1 mark)

$\Rightarrow k=\dfrac{1}{3}$ (1 mark)

b $E(X)=\sum xP(X=x)=0\times\dfrac{k}{2}+0.5\times k+1\times k^2+1.5\times 2k^2+2\times\dfrac{k}{2}$ (1 mark)

$E(X)=\dfrac{k}{2}+k^2+3k^2+k$

$=4k^2+\dfrac{3k}{2}$ (1 mark)

$=4\left(\dfrac{1}{3}\right)^2+\dfrac{3}{2}\left(\dfrac{1}{3}\right)$

$=\dfrac{4}{9}+\dfrac{1}{2}$ (1 mark)

$=\dfrac{17}{18}$

c $P(X\geq 1.25)=2k^2+\dfrac{k}{2}$ (1 mark)

$=2\left(\dfrac{1}{3}\right)^2+\dfrac{1}{6}$ (1 mark)

$=\dfrac{2}{9}+\dfrac{1}{6}$

$= \frac{7}{18}$ (1 mark)

d $\text{Var}(X) = E(X^2) - [E(X)]^2$ (1 mark)

$= \left[0^2 \times \frac{1}{6} + \left(\frac{1}{2}\right)^2 \times \frac{1}{3} + 1^2 \times \frac{1}{9} + \left(\frac{3}{2}\right)^2 \times \frac{2}{9} + 2^2 \times \frac{1}{6}\right] - \left(\frac{17}{18}\right)^2$ (1 mark)

$= 0.469$ (1 mark)

14 a $X \sim B(24, 0.04)$

$P(X = 2) = \binom{24}{2}(0.04)^2(0.96)^{22}$ (1 mark)

$= 0.180$ (1 mark)

b $P(X \le 4) = 0.998$ (2 marks)

c $P(X \ge 2) = 0.249$ (2 marks)

d $\text{Var}(X) = np(1-p)$ (1 mark)

$= 24 \times 0.04 \times 0.96$

$= 0.922$ (1 mark)

15 a $X \sim N(36, 3.12^2)$

$P(X > 40) = P\left(Z > \frac{40-36}{3.12}\right)$ (1 mark)

$= 0.1$ (1 mark)

b $P(34 < X < 38) = P\left(\frac{34-36}{3.12} < Z < \frac{38-36}{3.12}\right)$ (1 mark)

$= P(-0.641 < Z < 0.641)$ (1 mark)

$= P(Z < 0.641) - P(Z < -0.641)$ (1 mark)

$= 0.739 - 0.261$

$= 0.478$ (1 mark)

c $P\left(Z > \frac{M-36}{3.12}\right) = 0.015$ (1 mark)

$P\left(Z < \frac{M-36}{3.12}\right) = 0.985$

$\frac{M-36}{3.12} = 2.170$ (1 mark)

$\Rightarrow M = 42.77$

$\Rightarrow M = 42 \text{ minutes, } 46 \text{ seconds}$ (1 mark)

d $P(X < 30) = P\left(Z < \frac{30-36}{3.12}\right)$ (1 mark)

$= P(Z < -1.923)$

$= 0.027$ (1 mark)

$195 \times 0.027 = 5.3$ (1 mark)

Therefore, the expected number of days is 5 . (1 mark)

16 a Let X be the discrete random variable 'mass of a can of baked beans'.

Then $X \sim N(415, 12^2)$

$P(X > m) = 0.65 \Rightarrow P(X \leq m) = 1 - 0.65 = 0.35$ (1 mark)

Using inverse normal distribution on GDC $\Rightarrow m = 410.4$ (2 marks)

b You require $P(X > 422.5 \mid X > 420)$. (1 mark)

$$P(X > 422.5 \mid X > 420) = \frac{P(X > 422.5)}{P(X > 420)}$$

$$= \frac{1 - P(X \leq 422.5)}{1 - P(X \leq 420)}$$ (1 mark)

$$= \frac{0.266}{0.338}$$ (1 mark)

$= 0.787$ (1 mark)

c Using GDC

$P(X < 413.5) = 0.450$ (2 marks)

Let Y be the random variable 'Number of cans of beans having a mass less than 413.5 g'.

In Ashok's experiment, Y is Binomially distributed across 144 trials with probability of 'success' (i.e. mass less than 413.5 g) being 0.450.

So $Y \sim B(144, 0.450)$ (1 mark)

$P(Y \geq 75) = 0.0524$ (1 mark)

17 a $k\int_0^1 (10x^2 - x^3)\ dx = 1$ (1 mark)

$$k\left[\frac{10x^3}{3} - \frac{x^4}{4}\right]_0^1 = 1$$ (1 mark)

$$k\left(\frac{10}{3} - \frac{1}{4}\right) = 1$$ (1 mark)

$$\frac{37k}{12} = 1$$ (1 mark)

$$k = \frac{12}{37}$$

b $E(X)=\int_0^1 xf(x)\ \mathrm{d}x=\frac{12}{37}\int_0^1\left(10x^3-x^4\right)\ \mathrm{d}x$ (1 mark)

$=\frac{12}{37}\left[\frac{10x^4}{4}-\frac{x^5}{5}\right]_0^1$ (1 mark)

$=\frac{12}{37}\left(\frac{5}{2}-\frac{1}{5}\right)$ (1 mark)

$=\frac{276}{370}\left(=\frac{138}{185}\right)(=0.746)$ (1 mark)

c $E(X^2)=\int_0^1 x^2f(x)\ \mathrm{d}x=\frac{12}{37}\int_0^1\left(10x^4-x^5\right)\ \mathrm{d}x$ (1 mark)

$=\frac{12}{37}\left[2x^5-\frac{x^6}{6}\right]_0^1$ (1 mark)

$=\frac{12}{37}\left(2-\frac{1}{6}\right)$

$=\frac{22}{37}$ (1 mark)

$\mathrm{Var}(X)=E(X^2)-\left[E(X)\right]^2$ (1 mark)

$=\frac{22}{37}-\left(\frac{138}{185}\right)^2$

$=0.0382$ (1 mark)

d $\frac{12}{37}\int_0^m\left(10x^2-x^3\right)\ \mathrm{d}x=\frac{1}{2}$ (1 mark)

$\frac{12}{37}\left[\frac{10x^3}{3}-\frac{x^4}{4}\right]_0^m=\frac{1}{2}$ (1 mark)

$\frac{12}{37}\left(\frac{10m^3}{3}-\frac{m^4}{4}\right)=\frac{1}{2}$ (1 mark)

$\frac{10m^3}{3}-\frac{m^4}{4}=\frac{37}{24}$

$80m^3-6m^4=37$

GDC $\Rightarrow m=0.789$ (2 marks)

18 a $P\left(Z<\frac{110-\mu}{\sigma}\right)=0.10\Rightarrow\frac{110-\mu}{\sigma}=-1.282$ (2 marks)

$P\left(Z>\frac{130-\mu}{\sigma}\right)=0.45\Rightarrow\frac{130-\mu}{\sigma}=0.126$ (2 marks)

Attempt to solve simultaneously: (1 mark)

$\mu = 128$ (1 mark)

$\sigma = 14.2$ (1 mark)

b $P(|X - \mu|) < 0.22$

$0.5 - \dfrac{0.22}{2} = 0.39$ (1 mark)

$P(X < a) = 0.39 \Rightarrow a = 124.2$ (1 mark)

$P(X > b) = 0.39 \Rightarrow b = 132.2$ (1 mark)

So $124.2 < X < 132.2$ (1 mark)

19 a $k\int_0^{\frac{\pi}{2}} \cos x \; dx = 1$ (1 mark)

$k[\sin x]_0^{\frac{\pi}{2}} = 1$ (1 mark)

$k\left(\sin\dfrac{\pi}{2} - \sin 0\right) = 1$ (1 mark)

$k(1-0) = 1$ (1 mark)

$k = 1$

b $\int_0^{\frac{\pi}{2}} x\cos x \; dx = [x\sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \; dx$ (2 marks)

$\int_0^{\frac{\pi}{2}} x\cos x \; dx = [x\sin x]_0^{\frac{\pi}{2}} + [\cos x]_0^{\frac{\pi}{2}}$ (1 mark)

$= \left(\dfrac{\pi}{2} - 0\right) + (0-1)$ (1 mark)

$= \dfrac{\pi}{2} - 1$ (1 mark)

$\int_0^{\frac{\pi}{2}} x^2\cos x \; dx = [x^2\sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2x\sin x \; dx$ (2 marks)

$= [x^2\sin x]_0^{\frac{\pi}{2}} - 2[-x\cos x + \sin x]_0^{\frac{\pi}{2}}$ (1 mark)

$= [x^2\sin x - 2\sin x + 2x\cos x]_0^{\frac{\pi}{2}}$

$= \dfrac{\pi^2}{4} - 2$ (1 mark)

$\mathrm{Var}(X) = \int_0^{\frac{\pi}{2}} x^2\cos x \; dx - \left(\int_0^{\frac{\pi}{2}} x\cos x \; dx\right)^2$ (1 mark)

$= \dfrac{\pi^2}{4} - 2 - \left(\dfrac{\pi}{2} - 1\right)^2$ (1 mark)

$= \frac{\pi^2}{4} - 2 - \left(\frac{\pi^2}{4} - \pi + 1\right)$ (1 mark)

$= \pi - 3$